Recent Research in Science and Technology 2012, 4(9): 19-20
ISSN: 2076-5061
Available Online: http://recent-science.com/

A note on convergence of wilkinson and hilbert matrix with pre conditioned conjugate gradient method

Vinay Saxena*

*Department of Mathematics, Kisan PG College, Bahraich, (UP) 271801, India

Abstract

The present work is an attempt to make a study of Pre Conditioned Conjugate Gradient Method for almost Ill-conditioned matrices with computational sensitivity. Such matrices play very important role in solving physical problems arising in engineering and science. It is well known that the ill conditioning of a matrix system plays quite a dominant role adversely affect the performance of technique and the accuracy of the solution. In order to test validation of programmes 4 by 4 Wilkinson Matrix and 10 by 10 Hilbert Matrix are tested. Various computational aspects like as CPU time, number of iterations and convergence criteria are analyzed for this numerical solver. It is observed that Pre Conditioned Conjugate Gradient method is fast converging and gives very good results in both cases.

Keywords: Pre Conditioned Conjugate Gradient Method, 4 by 4 Wilkinson Matrix, 10 by 10 Hilbert Matrix

INTRODUCTION

The basic idea for Ill-conditioned matrices is "small change in input produces large changes in the output". In some situations, one can combat ill conditioning by transforming the problem into an equivalent set of equations that are not ill- conditioned. The efficiency of that particular scheme is related it relative amount of computation required for transformation, compare to the cost of doing calculations. The randomly generated matrices do not form a suitable set for testing sensitivity; almost all such matrices are very well behaved. The idea behind sensitivity analysis is very simple and quite general: Resolve the problem with slightly different data to see how sensitive the solution is to changes in the data. Different equation systems have been experimented in order to assess the viability of methods.

It is well known that when the matrix A is square and nonsingular, the Gaussian elimination process with iterative improvement will give an accurate answer unless the matrix is too ill-conditioned. But when the matrix is sparse and of large order, this method may be less efficient than iterative methods like Gauss-Seidal and Jacobi processes. These iterative methods do not always converge. In any case, efficient algorithms have seldom been reported for singular cases. Tanabe[1] investigated Projection method due to Kaczmarz for both Singular and non-singular systems and it determines the affine space formed by the solutions if they exist. Stewart[2] described a general class of algorithms for solving the equation AX = B, where A is nonsingular matrix. The errors in the iterates were characterized in terms of projectors construct able from the conjugate directions. The natural relations of the algorithms to well known matrix decomposition were pointed out. Saxena and Awasthi[3]described an efficient method for solving a system of linear equations with non-zero rows, whose coefficient matrix is almost singular or inconsistent. Saxena[4] discussed convergence of Wilkinson Matrix with Projection Method.

To the best of my knowledge, no user-friendly software is available for the ill- conditioned matrices with computational sensitivity. In order to test validation of programmes Wilkinson Matrix (deceptively simple, as it is already in lower triangular form, however it poses severe computational problem) and Hilbert Matrix (famous because it arise in many situations and it is so difficult to use in actual computations) are tested. Our concern is here to yield the solution of system of linear equations AX = B, if A is either 4 by 4 Wilkinson matrix or 10 by 10 Hilbert matrix. The various direct and iterative methods[5-6] for solving linear equations get failed or give solutions which are not good. It is investigated that various factorization methods are failed to solve this problem. Various computational aspects like as CPU time, number of iterations and convergence criteria are analyzed for the numerical solvers.

METHOD AND ALGORITHM

To solve the linear equation system, pre conditioned conjugate gradient call in each step for the evaluation of the matrix vector product AX and the solution of an auxiliary system MZ = r for a given right hand side vector "r". The matrix M is called pre conditioner and its choice is problem dependent. Some common, choice for M are the diagonal of A or an incomplete LU decomposition of A. The method of conjugate gradient works well on matrices that are either well conditioned or have just a few distinct eigen values. In this section, efforts are made to show how to pre condition a linear system so that the matrix of coefficients assumes one of these nice forms. The procedure for pre conditioned conjugate gradient method is as follows:

Start with the initial iteration, i.e , iteration number k=0 and initial guess vector X_0 and calculate the residual $r_0 = b - A X_0$; if it is zero, we have exact solution so proceeding up to while $r_k \neq 0$ or less than desired accuracy (here, accuracy is imposed over norm of vector X_k rather than individual vector).M is pre conditioned matrix

Received: July 02, 2012; Revised: Aug 20, 2012; Accepted: Sept 25, 2012.

*Corresponding Author
Vinay Saxena
Department of Mathematics, Kisan PG College, Bahraich, (UP) 271801, India

Tel: +91-9415178901; Fax: 91-5252-232824
Email: dr.vinaysaxena@gmail.com

which is chosen as diagonal matrix, whose inverse is very easy to compute and very less consumption of time to solve $MZ_k = r_k$ which is different from the directions as Z_k which is different from direction of steepest descent (i.e, r_k but giving better results).

Then calculate:

P_1 direction as $P_1 = Z_0$

Optimum step length for this direction by $\alpha_1 = \frac{r_0^T \; Z_0}{P_1^T \; A \; P_1}$

Improved vector $X_1 = X_0 + \alpha_1 P_1$ and $r_1 = r_0 - \alpha_1 A P_1$

For this r_1 again solve $MZ_k = r_k$.

Then the optimum step length for "P" direction as $\beta_k = \frac{r_{k-1}^T \; Z_{k-1}}{r_{k-2}^T \; Z_{k-2}}$

and direction as $P_k = Z_{k-1} + \beta_k P_{k-1}$

Using this direction calculate optimum step length in the direction " r" by $\alpha_k = \frac{r_{k-1}^T \; Z_{k-1}}{P_k^T \; A \; P_k}$

Improved vector as $X_k = X_{k-1} + \alpha_k P_k$ and direction r_k to get Z_k as $r_k = r_{k-1} - \alpha_k A P_k$

This process has to be continued until any one of the convergence criteria is satisfied-

(a) $\left|\frac{f(Xk+1) - f(Xk)}{f(Xk)}\right| \leq \varepsilon$

(b) $|Xk + 1 - Xk| \leq \varepsilon$

(c) $\frac{|Xk+1 - Xk|2}{|Xk+1|2} \leq \varepsilon$

Algorithm(Pre Conditioned Conjugate Gradient):

If $A \in R^{n\times n}$ is symmetric positive definite and $B \in R^n$, then the following algorithm solves the linear system AX = B using the method of conjugate gradients with pre conditioner $M \in R^{n\times n}$,here M is a diagonal matrix as m(i,j) = a(i,j)

(i) start k = 0;
$X_0 = 0$;
$r_0 = B$

(ii) if ($r_k \neq 0$)

(iii)k = k+1

(iv)if k =1

P(1) = Z(0)

else

$\beta_k = \frac{r_{k-1}^T \; Z_{k-1}}{r_{k-2}^T \; Z_{k-2}}$

$P_k = Z_{k-1} + \beta_k P_{k-1}$

(end of *if* statement)

(v) $\alpha_k = \frac{r_{k-1}^T \; Z_{k-1}}{P_k^T \; A \; P_k}$

(vi) $X_k = X_{k-1} + \alpha_k P_k$

(vii) $r_k = r_{k-1} - \alpha_k A P_k$

(viii) for i = 1,n

calculate eps = $\{\sum(x_i^{(k+1)} - x_i^{(k)})^2\} / \sum(x_i^{(k+1)})^2$

(ix) if eps is less than given accuracy

goto step(x)

else

goto step(ii)

(x) $X = X_k$

RESULTS AND DISCUSSION

The equation systems containing 4 by 4 Wilkinson matrix and 10 by 10 Hilbert matrix as coefficient matrix have been experimented in order to estimate the potential of Pre Conditioned Conjugate Gradient method. When AX = B, where A= 4 by 4 Wilkinson matrix is solved by Direct methods, it is observed that factorization methods do not give a good result. This is due to, in the step of traingulization the diagonal element of last row tends to zero. Therefore one of the factors goes to singular. Thus, perhaps all factorization methods are failed to solve this problem. When AX = B, where A= 10 by 10 Hilbert matrix is solved by Direct methods, we observed that elimination process with S.P.P. gives close result while factorization method failed. It is observed that among the iterative methods, Pre Conditioned Conjugate Gradient Method has been found to be fast converging and gives very good results in both cases.

REFERENCES

[1]. Tanbe K, 1971, Projection Method for solving a singular system of Linear equations and it's application, *Numer. Math,* 17,203-214

[2]. Stewart G W, 1973, Conjugate Direction Methods for solving systems of Linear Equations, *Numer. Math,* 21,285-297

[3]. Saxena V, Awasthi A K, 2012,Conjugate Gradient Method for ill – conditioned linear system, *Int. Multi Res J.*16-17,2(1).

[4]. Saxena V, 2010, A note on Convergence of Wilkinson Matrix with Projection Method, *Rec. Res. Sci Tech* 20-21,2(9)

[5]. S. D. Conte and Carl de Boor (1987) *Elementary Numerical Analysis,* McGRAW-HILL book company.

[6]. G. Golub and J.M.Ortega . *Scientific Computing An Introduction with Parallel Computing*. Academic Press, Inc.