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On weakly concircular ϕ- symmetric of $\in$-trans sasakian manifolds

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Abstract
The purpose of this paper is to introduce a new concept such as weakly concircular ϕ- symmetric $\in$-Trans-Sasakian manifold, and study its properties. A series of corollaries from the theorems are also obtained and a concrete example for the existence of such manifolds is provided.

Keywords: $\in$-almost contact metric manifold , $\in$-Trans-Sasakian manifold , weakly concircular ϕ-symmetric.

INTRODUCTION

In Mathematics, a weakly symmetric space is a notion introduced by the Norwegian Mathematician Atle Selberg in the 1950s as a generalization of a symmetric space, due to Elie Cartan. Geometrically the spaces are defined as complete Riemannian manifolds such that any two points can be exchanged by an isometry, the symmetric case being when the isometry is required to have period two. The classification of weakly symmetric spaces relies on that of periodic automorphism of complex bi semi simple Lie algebras.

In 2011, Shyamal Kumar Hui has studied the weak concircular Symmetries of trans-Sasakian manifolds. In 1989 Tamassy and Binh [9] have introduced the notion of Weakly Symmetric manifolds. In 1999 De and Bandyopadhyay [3] studied Weakly Symmetric manifolds and introduced following definition:

Definition 1.1: A non flat Riemannian manifold (M^n, g) (n>2) is called a weakly symmetric manifold if its curvature tensor R of type (0, 4) satisfies the condition

$$\begin{aligned}\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V)+B(Y)(X,Z,U,V)\\ &+B(Z)R(Y,X,U,V)+D(U)R(Y,Z,X,V)\\ &+D(V)R(Y,Z,U,X)\end{aligned} \tag{1.1}$$

for all vector fields X, Y, Z, U, V$\in \chi(M^n)$; $\chi(M)$ being the Lie algebra of smooth vector fields of M, where A,B, and D are 1-forms and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [11]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by [11]

$$\tilde{C}(Y,Z,U,V)=R(Y,Z,U,V)-\frac{r}{n(n-1)}[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)] \tag{1.2}$$

where r is the scalar curvature of the manifold.

Let $\{e_i : i=1,2,\dots\dots\dots,n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y,V)=\sum_{i=1}^{n}\tilde{C}(Y,e_i,e_i,V),$$

Then from (1.2), we get

$$P(Y,V)=S(Y,V)-\frac{r}{n}g(Y,V). \tag{1.3}$$

The tensor P is called the concircular Ricci symmetric tensor [4], which is a symmetric tensor of type (0, 2).

In 2009, Shaikh and Hui [7] have introduced the following definition:

Definition 1.2: A Riemannian manifold (M^n, g) (n>2) is called weakly concircular symmetric manifold if its concircular curvature tensor $\tilde{C}$,of type (0, 4) is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_X \tilde{C})(Y,Z,U,V) &= A(X)\tilde{C}(Y,Z,U,V)+B(Y)\tilde{C}(X,Z,U,V)\\ &+B(Z)\tilde{C}(Y,X,U,V)+D(U)\tilde{C}(Y,Z,X,V)\\ &+D(V)\tilde{C}(Y,Z,U,X)\end{aligned} \tag{1.4}$$

where A, B, D are 1-forms an n-dimensional manifold of this kind is denoted by $(W\tilde{C}S)_n$.

In [13], A.Bejancu and K. L. Duggal introduced the notion of $\in$-Sasakian manifolds with indefinite metric. In 1998, Xu Xufeng and Chao Xiaoli proved that every $\in$-Sasakian manifold is a hyper surface of an indefinite Kaehlerian manifold and established a necessary and sufficient condition for an odd dimensional Riemannian manifold to be an $\in$-Sasakian manifolds [14].In [15], U. C. De and Avijit Sarkar introduced and studied the notion of $\in$-Kenmotsu manifolds with indefinite metric giving an example.

The purpose of this paper is to introduce a new concept such as weakly concircular ϕ-symmetric $\in$-Trans-Sasakian manifold, and

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study its some properties. Section 2 is devoted to the preliminary results of $\in$-Trans-Sasakian manifolds that are needed in the rest of the sections. Recently S. K. Hui [12] studied weak concircular Symmetries of Trans-Sasakian manifolds. However, in section 3 of the paper new definition for weakly concircular ϕ- symmetric $\in$-Trans-Sasakian manifold and properties of these manifolds are also studied. In section 4 we have provide a concrete example for the existence of weakly concircular ϕ- symmetric $\in$-Trans-Sasakian manifold.

Preliminaries

In this section, we list the basic definitions and known results of $\in$-Trans-Sasakian manifolds.

Definition 2.1. [16] A (2n+1)-dimensional differentiable manifold (M, g) is said to be an $\in$-almost contact metric manifold, if it admits a (1, 1) tensor field ϕ, a structure vector field ξ, a 1-form η an indefinite metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi,\xi) = \in, \eta(X) = \in g(X,\xi) \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \in \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X,Y on M, where $\in$ is 1 or -1 according as ξ is space like or time like and rank ϕ is 2n.
From the above equations, one can deduce that

$$\phi\xi = 0, \eta(\phi X) = 0$$

Definition 2.2 An $\in$-almost contact metric manifold is called an $\in$-Trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi - \in \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \in \eta(Y)\phi X\}, \tag{2.4}$$

for any X, Y on M, where ∇ is Live-Civita connection with respect to g.

We note that if $\in$=1, i.e. structure vector field ξ is space like, and then an $\in$-Trans –Sasakian manifold is usual trans-Sasakian manifold [5].

A Trans-Sasakian manifold of type (0, 0), (0,β), (α, 0) are the cosympletic, β-Kenmotsu and α-Sasakian manifolds respectively. In particular if α=1,β=0, and α=0,β=1, then trans-Sasakian manifold reduces to Sasakian and Kenmotsu manifolds respectively.
For $\in$-Trans –Sasakian manifold, we have [17]

$$(\nabla_X \xi) = \in \{-\alpha\phi X + \beta(X - \eta(X)\xi\} \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta\{g(X,Y) - \in \eta(X)\eta(Y)\} \tag{2.6}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\}$$
$$+ \in \{(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y\} \tag{2.7}$$

$$\eta(R(X,Y)Z) = \in (\alpha^2 - \beta^2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}$$
$$+2 \in \alpha\beta\{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\}$$
$$+\{(X\beta)g(\phi^2 Y, Z) - (Y\beta)g(\phi^2 X, Z)\} + \{(X\alpha)g(\phi Y, Z) - (Y\alpha)g(\phi X, Z)\} \tag{2.8}$$

$$S(X,\xi) = \{2n(\alpha^2 - \beta^2) - \in (\xi\beta)\}\eta(X) - \in (\phi X)\alpha - \in (2n-1)(X\beta) \tag{2.9}$$

$$R(\xi, X)\xi = \{\alpha^2 - \beta^2 - \in (\xi\beta)\}\{-X + \eta(X)\xi\} - \{2\alpha\beta + \in (\xi\alpha)\}(\phi X) \tag{2.10}$$

$$S(\xi,\xi) = 2n\{\alpha^2 - \beta^2 - \in (\xi\beta)\} \tag{2.11}$$

$$2\alpha\beta + \in (\xi\alpha) = 0 \tag{2.12}$$

where R is the curvature tensor of type (1, 3) of the manifold and S is Ricci tensor.

Weakly Concircular ϕ symmetric $\in$-Trans-Sasakian manifolds

The notion of a weakly symmetric manifold was introduced by L. Tamassy and T. Q. Binh. Such a manifold have been studied by T. Q. Binh, M, Prvanovic and U. C. De and S. Bandyopadhyay.

Definition 3.1: A non-flat Riemannian manifold (M^n, g) (n>2) is called weakly symmetric if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W + C(Z)R(Y,X)W$$
$$+D(W)R(Y,Z)X + g(R(Y,Z)W, X)\rho, \tag{3.1}$$

where ∇ denotes the Levi-Civita connection on (M^n, g) and A, B, C, D and ρ are 1-forms and a vector fields respectively which are non zero simultaneously, and manifold is called weakly concircular symmetric if the concircular curvature tensor $\tilde{C}$ given by (1.2) satisfies the relation (3.1).Such manifolds has been denoted by $(WS)_n$ and $(WCS)_n$. In 1999 De and Bandyopadhyay proved the existence of a weakly symmetric manifold by an example. It was proved in 1995 by M Prvnovic that 1-forms and vector field must be related as follows

B(X)=C(X)=D(X), g(X, ρ)=D(X), $\forall$X, That is the weakly symmetric manifold is characterized by the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + D(Y)R(X,Z)W + D(Z)R(Y,X)W$$
$$+D(W)R(Y,Z)X + g(R(Y,Z)W, X)\rho, \tag{3.2}$$

g(X, ρ)=D(X), for all X.

The 1-forms A and D are the first and the second associated 1-forms respectively and manifold is called weakly concircular symmetric if the concircular curvature tensor $\tilde{C}$ given by (1.2) satisfies the relation (3.2).
In this paper we have introduced the following definition:

Definition 3.2: A non-flat Riemannian manifold (M^n, g) is called weakly concircular ϕ-symmetric if its concircular curvature tensor $\tilde{C}$ of type (1, 3) satisfies the condition:

$$\phi^2(\nabla_X \tilde{C})(Y,Z)W = A(X)\tilde{C}(Y,Z)W + D(Y)\tilde{C}(X,Z)W + D(Z)\tilde{C}(Y,X)W$$

$$+D(W)\tilde{C}(Y,Z)X+\tilde{C}(Y,Z,W,X)\rho \tag{3.3}$$

for all vector fields X, Y, Z, W $\in \chi(M^n);\chi(M)$ and A, D are associated 1-forms of the manifold and, $\tilde{C}$ (Y, Z, W, X)=g($\tilde{C}$ (Y, Z)W, X).

By observing definition 3.1, it is quite natural to think that this definition may be extended to $\in$-Trans-Sasakian manifolds. Thus we have the following definition.

Definition 3.3: A $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is said to be weakly concircular ϕ symmetric if its curvature tensor $\tilde{C}$ satisfies the condition (3.3).

Lemma 3.1: If a weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the relation

$$A(\xi)+2D(\xi)=\frac{2n^2\{2\beta(\xi\beta)+\in\xi(\xi\beta)-2\alpha(\xi\alpha)\}+\in dr(\xi)}{2n^2\{\alpha^2-\in(\xi\beta)-\beta^2\}-\in r} \tag{3.4}$$

holds.

Proof: Operating by g on both sides of (3.3) and substituting for ϕ^2 in (3.3),

$$g\{\phi^2(\nabla_X\tilde{C}(Y,Z)W,U\}=A(X)g(\tilde{C}(Y,Z)W,U)+D(Y)g(\tilde{C}(X,Z)W,U)$$
$$+D(Z)g(\tilde{C}(Y,X)W,U)+D(W)g(\tilde{C}(Y,Z)X,U)+D(U)g(\tilde{C}(Y,Z)W,X). \tag{3.5}$$

By virtue of (2.1), relation (3.5) can be expressed as

$$-g\{(\nabla_X\tilde{C})(Y,Z)W,U\}+\eta\{(\nabla_X\tilde{C}(Y,Z)W\}g(U,\xi)$$
$$=A(X)g(\tilde{C}(Y,Z)W,U)+D(Y)g(\tilde{C}(X,Z)W,U)$$
$$+D(Z)g(\tilde{C}(Y,X)W,U)+D(W)g(\tilde{C}(Y,Z)X,U)$$
$$+D(U)g(\tilde{C}(Y,Z)W,X) \tag{3.6}$$

If ξ is orthogonal to U then η(U) =0=g (U,ξ).

Setting Y=U= e_i for i= 1, 2, 3…..2n+1, where {e_i} are orthonormal basis of the tangent space at each point P of the manifold and taking the sum from i=1 to i=2n+1 it is easy to see that the relation (3.6) reduces to

$$-(\nabla_X S)(Z,W)+\frac{dr(X)}{n}g(Z,W)$$
$$=A(X)[S(Z,W)-\frac{r}{n}g(Z,W)]+D(Z)[S(X,W)-\frac{r}{n}g(X,W)]$$
$$+D(W)[S(Z,X)-\frac{r}{n}g(Z,X)]+D(R(X,Z)W)+D(R(X,W)Z)$$
$$-\frac{r}{n(n-1)}[2D(X)g(Z,W)-D(Z)g(X,W)-D(W)g(X,Z)] \tag{3.7}$$

where S is the Ricci curvature tensor of the type (0, 2). Putting X=Z=W=ξ in (3.7), we get

$$-(\nabla_\xi S)(\xi,\xi)+\frac{dr(\xi)}{n}g(\xi,\xi)$$
$$=A(\xi)[S(\xi,\xi)-\frac{r}{n}g(\xi,\xi)]+D(\xi)[S(\xi,\xi)-\frac{r}{n}g(\xi,\xi)]$$
$$+D(\xi)[S(\xi,\xi)-\frac{r}{n}g(\xi,\xi)]+D(R(\xi,\xi)\xi)+D(R(\xi,\xi)\xi) \tag{3.8}$$

Now substituting for g(ξ, ξ), R(ξ, ξ)ξ, and S(ξ, ξ), from (2.2), (2.10), and (2.11) respectively in (3.8), and after simplification the result (3.4) of the theorem 3.1 follows. This completes the proof of the theorem.

Remark: for weakly concircular ϕ symmetric $\in$-Trans-Sasakian manifold of type (1, 0) and (0, 1) (i.e. weakly concircular ϕ symmetric Sasakian and Kenmotsu manifolds), it is easy to see from (3.4) of Theorem 3.1 that

$$A(\xi)+2D(\xi)=-\frac{dr(\xi)}{r}$$

Theorem 3.2: If a weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the associated 1-from D is given by

$$D(W)=\frac{[2n\{2\beta(\xi\beta)-2\alpha(\xi\alpha)\}+\in\xi(\xi\beta)+\frac{dr(\xi)}{\in n}]\eta(W)}{(2n-1)[\alpha^2-\beta^2-\in(\xi\beta)]-\frac{\in(n-2)r}{n(n-1)}}$$
$$+\frac{(2n-1)\in W(\xi\beta)+\in(\phi W)(\xi\alpha)}{(2n-1)[\alpha^2-\beta^2-\in(\xi\beta)]-\frac{\in(n-2)r}{n(n-1)}}$$

$$+D(\xi)[\frac{\{(2n-1)(\alpha^2-\beta^2)-\frac{(n-2)r}{n(n-1)\in}\}\eta(W)-\in(2n-1)(W\beta)-\in(\phi W)\alpha}{(2n-1)[\alpha^2-\beta^2-\in(\xi\beta)]-\frac{\in(n-2)r}{n(n-1)}}]$$

$$-\frac{2n\{2\beta(\xi\beta)+\in\xi(\xi\beta)-2\alpha(\xi\alpha)\}+\frac{\in dr(\xi)}{n}}{[2n\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in r}{n}][(2n-1)\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in(n-2)r}{n(n-1)}]}$$

$$[\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\frac{r}{\in n}\}\eta(W)-\in(2n-1)(W\beta)-\in(\phi W)\alpha]. \tag{3.9}$$

for any vector field W.

Proof: Taking X=Z=ξ in (3.7), we get

$$-(\nabla_\xi S)(\xi,W)+\frac{dr(\xi)}{n}g(\xi,W)$$
$$=A(\xi)[S(\xi,W)-\frac{r}{n}g(\xi,W)]+D(\xi)[S(\xi,W)-\frac{r}{n}g(\xi,W)]$$
$$+D(W)[S(\xi,\xi)-\frac{r}{n}g(\xi,\xi)]+D(R(\xi,\xi)W)+D(R(\xi,W)\xi)$$
$$-\frac{r}{n(n-1)}[D(\xi)g(\xi,W)-D(W)g(\xi,\xi)] \tag{3.10}$$

now substituting for g(ξ, W), R(ξ, W)ξ, S(ξ, ξ), from (2.2), (2.10) and (2.11) respectively in (3.10), we get

$$-(\nabla_\xi S)(\xi,W)+\frac{dr(\xi)}{\in n}\eta(W)$$
$$=(A(\xi)+D(\xi))[S(\xi,W)-\frac{r}{\in n}\eta(W)]$$
$$+D(W)[(2n-1)\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in(n-2)r}{n(n-1)}]$$

$$+[\alpha^2-\in(\xi\beta)-\beta^2-\frac{r}{\in n(n-1)}]\eta(W)D(\xi) \quad (3.11)$$

from (2.9), we have

$$(\nabla_\xi S)(\xi,W)=\nabla_\xi S(\xi,W)-S(\nabla_\xi\xi,W)-S(\xi,\nabla_\xi W)$$

$$=\nabla_\xi S(\xi,W)-S(\xi,\nabla_\xi W)$$

$$=[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-\in\xi(\xi\beta)]\eta(W)$$

$$-(2n-1)\in W(\xi\beta)-\in(\phi W)(\xi\alpha) , \quad (3.12)$$

Substituting (3.12) in (3.11), we get

$$[2n\{2\beta(\xi\beta)-2\alpha(\xi\alpha)\}+\in\xi(\xi\beta)+\frac{dr(\xi)}{\in n}]\eta(W)+\in\{(\phi W)(\xi\alpha)+(2n-1)W(\xi\beta)\}$$

$$=(A(\xi)+D(\xi))[S(\xi,W)-\frac{r}{\in n}\eta(W)]$$

$$+D(W)[(2n-1)\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in(n-2)r}{n(n-1)}]$$

$$+[\alpha^2-\in(\xi\beta)-\beta^2-\frac{r}{\in n(n-1)}]\eta(W)D(\xi) \quad (3.13)$$

Now by using (2.9), and substituting for A(ξ)+2D(ξ) from (3.4) in (3.13), and after simplifying for D(W), this completes the proof of the Theorem 3.2.

Theorem 3.3: If a weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the associated 1-from A and D, the following relation

$$A(X)+2D(X)=\frac{2n[2\beta(X\beta)+\in X(\xi\beta)-2\alpha(X\alpha)]}{[2n(\alpha^2-\in(\xi\beta)-\beta^2)-\frac{\in r}{n}]}$$

$$+\frac{2\alpha[\eta(X)(\xi\alpha)+(2n-1)(\phi X)\beta-(X\alpha)]}{[2n(\alpha^2-\in(\xi\beta)-\beta^2)-\frac{\in r}{n}]}$$

$$-\frac{2\beta[(\phi X)\alpha+(2n-1)\{X\beta-(\xi\beta)\eta(X)\}]-\frac{\in dr(X)}{n}}{[2n(\alpha^2-\in(\xi\beta)-\beta^2)-\frac{\in r}{n}]}$$

$$+\frac{2[2n\{2\beta(\xi\beta)-2\alpha(\xi\alpha)\}+\in\xi(\xi\beta)+\frac{dr(\xi)}{\in n}]\eta(X)}{[2n(\alpha^2-\in(\xi\beta)-\beta^2)-\frac{\in r}{n}]}$$

$$+\frac{2[(2n-1)\in X(\xi\beta)+\in(\phi X)\xi\alpha]}{[2n(\alpha^2-\in(\xi\beta)-\beta^2)-\frac{\in r}{n}]}$$

$$-\frac{2[2n\{2\beta(\xi\beta)+\in\xi(\xi\beta)-2\alpha(\xi\alpha)\}+\frac{\in dr(\xi)}{n}]}{[2n(\alpha^2-\in(\xi\beta)-\beta^2)-\frac{\in r}{n}]^2}$$

$$[\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\frac{r}{\in n}\}\eta(X)-\in(2n-1)(X\beta)-\in(\phi X)\alpha] \quad (3.14)$$

holds.

Proof: Substituting Z=W=ξ in (3.7), we get

$$-(\nabla_X S)(\xi,\xi)+\frac{dr(X)}{n}g(\xi,\xi)$$

$$=A(X)[S(\xi,\xi)-\frac{r}{n}g(\xi,\xi)]+2D(\xi)[S(X,\xi)-\frac{r}{n}g(X,\xi)]$$

$$+2D(R(X,\xi)\xi)-\frac{r}{n(n-1)}[2D(X)g(\xi,\xi)-2D(\xi)g(X,\xi)] \quad (3.15)$$

Now substituting for g(ξ,ξ), S(ξ, ξ), R(X, ξ)ξ, from (2.2), (2.11), and (2.10) respectively in (3.15), we get

$$-(\nabla_X S)(\xi,\xi)+\frac{\in dr(X)}{n}$$

$$=[2n\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in r}{n}]A(X)$$

$$+2D(X)[\alpha^2-\in(\xi\beta)-\beta^2-\frac{\in r}{n(n-1)}]$$

$$+2D(\xi)[\{(2n-1)(\alpha^2-\beta^2)-\frac{n-2}{\in n(n-1)}r\}\eta(X)-\in\{(\phi X)\alpha+(2n-1)(X\beta)] \quad (3.16)$$

Now we have

$$(\nabla_X S)(\xi,\xi)=\nabla_X S(\xi,\xi)-2S(\nabla_X\xi,\xi),$$

which yields by using (2.5) and (2.11) that

$$(\nabla_X S)(\xi,\xi)=2n\{2\alpha(X\alpha)-2\beta(X\beta)-\in X(\xi\beta)\}$$

$$+2\alpha[(X\alpha)-\eta(X)(\xi\alpha)-(2n-1)(\phi X)\beta]$$

$$+2\beta[(\phi X)\alpha+(2n-1)\{X\beta-(\xi\beta)\eta(X)\}] \quad (3.17)$$

$$D(\xi)[\{(2n-1)(\alpha^2-\beta^2)-\frac{n-2}{\in n(n-1)}r\}\eta(X)-\in\{(\phi X)\alpha+(2n-1)(X\beta)]$$

$$=D(X)[(2n-1)\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in(n-2)r}{n(n-1)}]$$

$$-[2n\{2\beta(\xi\beta)-2\alpha(\xi\alpha)\}+\in\xi(\xi\beta)+\frac{dr(\xi)}{\in n}]\eta(X)$$

$$-[\in(2n-1)X(\xi\beta)+\in(\phi X)(\xi\alpha)]$$

$$+\frac{2n\{2\beta(\xi\beta)+\in\xi(\xi\beta)-2\alpha(\xi\alpha)\}+\frac{\in dr(\xi)}{n}}{[2n\{\alpha^2-\beta^2-\in(\xi\beta)\}-\frac{\in r}{n}]}$$

$$[\{2n(\alpha^2-\beta^2)-\in(\xi\beta)-\frac{r}{\in n}\}\eta(X)-\in(2n-1)(X\beta)-\in(\phi X)\alpha]. \quad (3.18)$$

Using (3.17) and (3.18) in (3.16) and after simplifying, we get (3.14). This completes the proof of Theorem 3.3.
In particular if, ϕgradα=gradβ, then,

$\xi\beta$=g (ξ, gradβ)

=g (ξ, ϕgradα)

= $\eta(\phi$grad$\alpha)$

=0

Substituting this value of $\xi\beta$ in (3.14)

$$A(X)+2D(X)=\frac{2n[2\beta(X\beta)-2\alpha(X\alpha)]}{[2n(\alpha^2-\beta^2)-\frac{\in r}{n}]}+\frac{2\alpha[\eta(X)(\xi\alpha)+(2n-1)(\phi X)\beta-(X\alpha)]}{[2n(\alpha^2-\beta^2)-\frac{\in r}{n}]}$$

$$-\frac{2\beta[(\phi X)\alpha+(2n-1)\{X\beta\}]-\frac{\in dr(X)}{n}}{[2n(\alpha^2-\beta^2)-\frac{\in r}{n}]}+\frac{2[4n\{-\alpha(\xi\alpha)\}+\frac{dr(\xi)}{\in n}]\eta(X)}{[2n(\alpha^2-\beta^2)-\frac{\in r}{n}]}$$

$$+\frac{2[\in(\phi X)\xi\alpha]}{[2n(\alpha^2-\beta^2)-\frac{\in r}{n}]}-\frac{2[4n\{-\alpha(\xi\alpha)\}+\frac{\in dr(\xi)}{n}]}{[2n(\alpha^2-\beta^2)-\frac{\in r}{n}]^2}$$

$$[\{2n(\alpha^2-\beta^2)-\frac{r}{\in n}\}\eta(X)-\in(2n-1)(X\beta)-\in(\phi X)\alpha] \tag{3.19}$$

Corollary 3.1: If a weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, satisfies the condition ϕgradα=gradβ, then the relation (3.19) holds.
If β =0 and α=1, then (3.19) yields.

$$A(X)+2D(X)+\frac{\in dr(X)}{2n^2-\in r}=0 \tag{3.20}$$

Corollary 3.2: If a weakly concircular $\in$-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the relation (3.20) holds.

Corollary 3.3: If a weakly concircular $\in$-α-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the relation

$$A(X)+2D(X)=\frac{2n[-2\alpha(X\alpha)]}{[2n(\alpha^2)-\frac{\in r}{n}]}+\frac{2\alpha[\eta(X)(\xi\alpha)-(X\alpha)]}{[2n(\alpha^2)-\frac{\in r}{n}]}-\frac{\frac{\in dr(X)}{n}}{[2n(\alpha^2)-\frac{\in r}{n}]}+\frac{2[2n\{-2\alpha(\xi\alpha)\}+\frac{dr(\xi)}{\in n}]\eta(X)}{[2n(\alpha^2)-\frac{\in r}{n}]}$$

$$+\frac{2[\in(\phi X)\xi\alpha]}{[2n(\alpha^2)-\frac{\in r}{n}]}-\frac{2[2n\{-2\alpha(\xi\alpha)\}+\frac{\in dr(\xi)}{n}]}{[2n(\alpha^2)-\frac{\in r}{n}]^2}$$

$$[\{2n(\alpha^2)-\frac{r}{\in n}\}\eta(X)-\in(\phi X)\alpha] \tag{3.21}$$

holds.

Proof: For the weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric of type (α, 0) i.e.; for α-Sasakian manifold, β=0 and $\alpha\neq0$ so that (3.19) after simplification yields (3.21), hence the proof of corollary 3.3 completes.

Corollary 3.4 If a weakly concircular $\in$-β-Kenmotsu manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the relation

$$A(X)+2D(X)=\frac{2n[2\beta(X\beta)]}{[2n(-\beta^2)-\frac{\in r}{n}]}-\frac{2\beta[((2n-1)\{X\beta\}]-\frac{\in dr(X)}{n}}{[2n(-\beta^2)-\frac{\in r}{n}]}+\frac{2[\frac{dr(\xi)}{\in n}]\eta(X)}{[2n(-\beta^2)-\frac{\in r}{n}]}$$

$$-\frac{2[\frac{\in dr(\xi)}{n}]}{[2n(-\beta^2)-\frac{\in r}{n}]^2}[\{2n(-\beta^2)-\frac{r}{\in n}\}\eta(X)-\in(2n-1)(X\beta)] \tag{3.22}$$

holds.

Proof: For the weakly concircular $\in$-Trans-Sasakian manifold (M^{2n+1},g) (n>1) is ϕ-symmetric of type (0,β) i.e.; for β-Kenmotsu manifold, $\beta\neq0$, and α=0 so that (3.19) after simplification yields (3.22), hence the proof of Corollary 3.4 completes.
If β =1 and α=0, then (3.19) yields.

$$A(X)+2D(X)-\frac{\in dr(X)}{2n^2+\in r}=0 \tag{3.23}$$

Corollary 3.2 If a weakly concircular $\in$-Kenmotsu manifold (M^{2n+1},g) (n>1) is ϕ-symmetric, then the relation (3.23) holds.

Example for Weakly Concircular ϕ symmetric $\in$-Trans-Sasakian manifolds

Example 4.1: Let us consider a 3-dimensional manifold M={(x, y, z)$\in$ R^3 : z≠0}, where (x, y, z) are the standard coordinates in R^3. Let $e_1=e^x(\frac{\partial}{\partial z}+y\frac{\partial}{\partial x}), e_2=e^x\frac{\partial}{\partial y}, e_3=\frac{\partial}{\partial x}$, which are linearly independent vector fields at each point of M. define a semi-Riemannian metric g on M as

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0, g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=\in,$$

where $\in$=±1.

Let η be the 1-form defined by $\eta(Z) = \in g(Z, e_3)$, for any $Z \in \Gamma(TM)$ and ϕ be the tensor field of type (1, 1) defined by $\phi e_1 = e_2, \phi e_2 = -e_1, \phi e_3 = 0$. Then by applying linearity of ϕ and g, we have

$$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3, g(\phi Z, \phi U) = g(Z, U) - \in \eta(Z)\eta(U),$$

for any $Z, U \in \Gamma(TM)$. Hence for $e_3 = \xi$, (ϕ, ξ, η, g, $\in$) defines an $\in$-almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of type (1, 3), then we have

$$[e_1, e_2] = \in (ye^x e_2 - e^{2x} e_3), [e_1, e_3] = - \in e_1, [e_2, e_3] = - \in e_2.$$

By using Koszul's formula for the Levi-Civita connection with respect to g, we obtain

$$\nabla_{e_1} e_3 = - \in e_1 + \frac{1}{2} \in e^{2x} e_2, \nabla_{e_2} e_3 = - \in e_2 - \frac{1}{2} \in e^{2x} e_1, \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = -\frac{1}{2} \in e^{2x} e_3, \nabla_{e_2} e_2 = \in e_3 + \in ye^x e_1, \nabla_{e_3} e_2 = -\frac{1}{2} \in e^{2x} e_1,$$

$$\nabla_{e_1} e_1 = \in e_3, \nabla_{e_2} e_1 = - \in ye^x e_2 + \frac{1}{2} \in e^{2x} e_3, \nabla_{e_3} e_1 = \frac{1}{2} \in e^{2x} e_2.$$

Now, for $e_3 = \xi$, above results satisfy

$$\nabla_X \xi = \in \{-\alpha \phi X + \beta(X - \eta(X)\xi)\},$$

with $\alpha = -\frac{1}{2} e^{2x}$ and $\beta = -1$. Consequently M (ϕ, ξ, η, g, $\in$) is a 3-dimensional $\in$-Trans-Sasakian manifold.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_1, e_2)e_2 = -(1 + \frac{3}{4} e^{4x} + y^2 e^{2x}) e_1, R(e_1, e_2)e_1 = (1 + \frac{3}{4} e^{4x} + y^2 e^{2x}) e_2$$

$$R(e_2, e_3)e_3 = -e^{2x} e_1 + (\frac{1}{4} e^{4x} - 1) e_2, R(e_1, e_3)e_3 = e^{2x} e_2 + (\frac{1}{4} e^{4x} - 1) e_1,$$

$$R(e_1, e_3)e_2 = -e^{2x} e_3, R(e_1, e_3)e_1 = -(1 + \frac{1}{4} e^{4x}) e_3,$$

$$R(e_2, e_3)e_2 = ye^x e_1 + (1 - \frac{1}{4} e^{4x}) e_3, R(e_3, e_1)e_1 = (1 + \frac{1}{4} e^{4x}) e_3,$$

$$R(e_3, e_2)e_1 = -e^{2x} e_3 + ye^x e_2.$$

and the components which can be obtained from these by the symmetry properties.

Using the components of the curvature tensor, and (1.2) we can easily calculate the concircular curvature tensor as follows:

$$\tilde{C}(e_1, e_2)e_2 = R(e_1, e_2)e_2 - \frac{r}{6}[g(e_2, e_2)e_1 - g(e_1, e_2)e_2]$$

$$= -(1 + \frac{3}{4} e^{4x} + y^2 e^{2x} + \frac{\in r}{6}) e_1$$

$$\tilde{C}(e_1, e_2)e_1 = (1 + \frac{3}{4} e^{4x} + y^2 e^{2x} + \frac{\in r}{6}) e_2,$$

$$\tilde{C}(e_3, e_2)e_1 = -e^{2x} e_3 + ye^x e_2,$$

$$\tilde{C}(e_2, e_3)e_3 = -e^{2x} e_1 + (\frac{1}{4} e^{4x} - 1 - \frac{\in r}{6}) e_2, \tilde{C}(e_1, e_3)e_3 = e^{2x} e_2 + (\frac{1}{4} e^{4x} - 1 - \frac{\in r}{6}) e_1,$$

$$\tilde{C}(e_1, e_3)e_2 = -e^{2x} e_3, \tilde{C}(e_1, e_3)e_1 = -(1 + \frac{1}{4} e^{4x} - \frac{\in r}{6}) e_3,$$

$$\tilde{C}(e_2, e_3)e_2 = ye^x e_1 + (1 - \frac{1}{4} e^{4x} + \frac{\in r}{6}) e_3, \tilde{C}(e_3, e_1)e_1 = (1 + \frac{1}{4} e^{4x} - \frac{\in r}{6}) e_3.$$

Let us conceder operating by g on both sides of (3.3), we get

$$g\{\phi^2(\nabla_X \tilde{C}(Y, Z)W, U\} = A(X)g(\tilde{C}(Y, Z)W, U) + D(Y)g(\tilde{C}(X, Z)W, U)$$

$$+ D(Z)g(\tilde{C}(Y, X)W, U) + D(W)g(\tilde{C}(Y, Z)X, U)$$

$$+ D(U)g(\tilde{C}(Y, Z)W, X) \tag{4.1}$$

$$g\{\phi^2(\nabla_{e_i} \tilde{C}(Y, Z)W, U\} = A(e_i)g(\tilde{C}(Y, Z)W, U) + D(Y)g(\tilde{C}(e_i, Z)W, U)$$

$$+ D(Z)g(\tilde{C}(Y, e_i)W, U) + D(W)g(\tilde{C}(Y, Z)e_i, U)$$

$$+ D(U)g(\tilde{C}(Y, Z)W, e_i)$$

where $e_i : i = 1, 2, 3,$ if R(Y, Z, W, U)≠0 then we can find from (4.1) the 1-forms as (4.2) and (4.3)

$$A(e_i) = \frac{g(\phi^2(\nabla_{e_i} \tilde{C})(Y, Z)W, U)}{\tilde{C}(Y, Z, W, U)}, i = 1, 2, 3.$$

and

$$D(Y)g(\tilde{C}(e_i, Z)W, U) + D(Z)g(\tilde{C}(Y, e_i)W, U)$$

$$+ D(W)g(\tilde{C}(Y, Z)e_i, U) + D(U)g(\tilde{C}(Y, Z)W, e_i) = 0$$

Any vector fields Y, Z, W, U can be written as

$$Y = b_1 e_1 + b_2 e_2 + b_3 e_3, Z = c_1 e_1 + c_2 e_2 + c_3 e_3, W = d_1 e_1 + d_2 e_2 + d_3 e_3,$$
$$U = f_1 e_1 + f_2 e_2 + f_3 e_3.$$

Substituting these vector fields in (4.4), we find

$$p_1 D(e_1) + q_1 D(e_2) + r_1 D(e_3) = 0$$

$$p_2 D(e_1) + q_2 D(e_2) + r_2 D(e_3) = 0$$

$$p_3 D(e_1) + q_3 D(e_2) + r_3 D(e_3) = 0$$

Where p_i, q_i, r_i for i=1, 2, 3 are the functions of x, y, z.
The set of equations in (4.3) are homogeneous in D, the trivial solution always exist, so that $D(e_i) = 0$, i=1, 2, 3. Thus one can state,

Theorem 4.1: A weakly concircular ϕ-symmetric $\in$-Trans-Sasakian manifold (M^3, g) (n>1), can be a ϕ-recurrent $\in$-Trans-Sasakian manifold.

If p_i, q_i, r_i for i=1, 2, 3 are such that

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

Then the system (4.3) has infinite number of solutions, hence one can state

Theorem 4.2: There exists a $\in$- Trans-Sasakian manifold which is weakly concircular ϕ symmetric but neither ϕ symmetric nor ϕ-recurrent.

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