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Comparison of Single Term Walsh Series Technique and Extended RK Methods Based on Variety of Means to Solve Stiff Non-linear Systems

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Abstract

This paper presents a comparison of Single Term Walsh Series (STWS) technique and the extended Runge-Kutta (RK) methods based on variety of means such as Arithmetic Mean (AM), Harmonic Mean (HaM), Centroidal Mean (CeM) and Contraharmonic Mean (CoM) to solve stiff non-linear systems of initial value problems (IVPs). Numerical solutions of some stiff non-linear systems are investigated for their stiffness. The discrete solutions obtained through STWS technique are compared with that of the RK methods based on variety of means. The applicability of the STWS technique has been demonstrated. The results show that the STWS technique is more suitable to solve stiff non-linear systems including highly stiff problems.

Key Words: STWS, WF, BPF, Stiffness Ratio, RK, AM, HaM, CeM, CoM.

Introduction

Many physical systems such as nuclear reactors and laser oscillators, etc. give rise to stiff non-linear ordinary differential equations (ODEs) in which the magnitudes of the eigenvalues vary greatly. Methods not designed for stiff problems are ineffective on intervals where the solution changes slowly because they use time steps small enough to resolve the fastest possible change. Stiff problems typically arise in chemical kinetics, nuclear reactor theory, control theory, biochemistry, climatology, electronics, fluid dynamics, etc.

Cash [1] has derived a class of extended backward differentiation formulae suitable for the approximate numerical integration of stiff system of first order ODEs. Lopez [2] has explained an explicit two-step method for solving stiff systems of ODEs. Ismail et al. [3] have suggested a new higher order predictor corrector method for solving stiff systems. Hsiao [4] has proposed Haar wavelet approach to linear stiff systems. Bujurke et al. [5] have obtained numerical solution of stiff systems from non-linear dynamics using Single Term Haar Wavelet Series.

The STWS technique was introduced by Rao et al. [6] using Walsh Functions (WFs). This is based on an approach called "single segment approximation" which avoids operational matrices of large size and maximizes the reduction in the computational effort. This method provides block-pulse and discrete solutions of problems, to any length of time, in an easy manner. Many researchers have made use of STWS technique to solve different systems such as time invariant and time varying singular systems, linear and non-linear singular systems, singularly perturbed systems, bilinear systems, optimal control of linear time invariant and time varying systems etc. [7 - 14].

RK methods are being used widely to solve many problems in Science and Engineering because of their efficiency, flexibility and accuracy. Yaakub and Evans [15] have studied RK methods based on different means to solve IVPs. Abdul Majid Wazwaz [16] has compared the modified third order RK formulae based on variety of means. Sanugi and Evans [17] have introduced the concept of fourth order RK formulae based on HaM.

This paper presents a comparison of the STWS technique and the fourth order extended RK methods based on variety of means AM, HaM, CeM and CoM to solve stiff non-linear systems. The effectiveness of the STWS method has been demonstrated by considering some examples of stiff non-linear systems.

Stiffness

Traditionally in numerical analysis, a **linear stiff system** of size n is defined by Lambert [18], Re(λ_i) < 0, 1 ≤ i ≤ n, where λ_i are the eigenvalues of the Jacobian of the system with $\max\limits_{1\le i\le n}\left|\mathrm{Re}(\lambda_i)\right| \gg \min\limits_{1\le i\le n}\left|\mathrm{Re}(\lambda_i)\right|$. The *Stiffness Ratio* (SR) provides a quantitative measure of stiffness:

$$SR = \frac{\max\limits_{1\le i\le n}\left|\mathrm{Re}(\lambda_i)\right|}{\min\limits_{1\le i\le n}\left|\mathrm{Re}(\lambda_i)\right|}. \qquad (2.1)$$

By this definition, a stiff problem has a stable fixed point with eigenvalues of greatly differing magnitudes; large negative eigenvalues correspond to fast decaying transients $e^{\lambda t}$ in the solution.

The definition of linear stiffness is not relevant for non-linear systems. The stiffness ratio defined by Eqn. (2.1) is often not a good measure of stiffness even for linear systems, since if the minimum eigenvalue is zero, the problem has infinite stiffness ratio. The standard form for first-order non-linear ODEs is as follows:

$y_i^{'} = f_i(y_1, y_2, ..., y_N)$ for i = 1, 2, . . . , N,

where N is the number of equations and take initial conditions $y_i(0) = \alpha_i$. It can be linearised around time $t=t_n$ using a Taylor expansion. Retaining only the first two terms

$y_i^{'} \approx f_i(y_n) + J(t_n)(y - y_n)$, *where*

$J(t_n) = \left\{ J_{ij} = \left[\frac{\partial f_i(y)}{\partial y_j} \right] t_n \right\}$ is the Jacobian matrix for the problem at $t = t_n$. Here, J_{ij} is the element of *J* in row *i* and column *j*.

The definition of stiffness utilizes the eigen values of the Jacobian matrix. Stiff ODEs are called extremely stable or super stable if there is at least one of the eigenvalues with a large negative real part [19].

Walsh Series

It is well known that a function, which is periodic, may be expanded into a Fourier series or Power series. In a similar manner, a function f(t), which is integrable in [0,1) may be approximated by using Walsh functions as

$$f(t) = \sum_{i=0}^{\infty} f_i \ \psi_i(t)$$

where $\psi_i(t)$ is the i^{th} Walsh function and f_i is the corresponding coefficient.

In practice, while approximating a function, only the first 'm' components are considered. If the coefficient of the Walsh functions are concisely written as 'm' vectors, then

$F = (f_0, f_1, \ldots, f_{m-1})^T$

and $\psi(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_{m-1}(t)]^T$

where $m = 2^k$, k is an integer and T denotes transpose. Then the above function f(t) becomes

$$f(t) \approx F^T \psi(t).$$

The coefficients f_i are chosen to minimise the mean integral square error

$$\in = \int_0^1 [f(t) - F^T \psi(t)]^2 dt.$$

The coefficients are given by

$$f_i = \int_0^1 f(t)\psi_i(t)dt.$$

It has been proved by Chen et al. (1975a) that

$$\int_0^t f(t)dt \approx F^T E \psi(t),$$

where E is called the operational matrix for integration in WF. In STWS, $E = \frac{1}{2}$.

STWS Technique for Non-linear Systems

Consider a non-linear system of the following form:

$\dot{x}(\tau) = f(t, x(t), u(t)),\ x(0) = x_0,$ (4.1)

where the non-linear function $f \in R^n$, the state $x(t) \in R^n$, and the control $u(t) \in R^q$.

With the STWS approach, the given function is expanded in the normalized interval $\tau \in [0, 1)$, which corresponds to *t* $\in [0, 1/m)$ by defining $\tau = mt$, *m* being any integer. Normalizing Eqn. (4.1) by defining $\tau = mt$, we get

$m\dot{x}(\tau) = f(\tau, x(\tau), u(\tau)),\ \ x(0) = x_0$

(4.2)

Let $\dot{x}(\tau)$ and $x(\tau)$ be expanded by STWS series in the k^{th} interval as

$\dot{x}(\tau) = V^{(k)}\psi_0(\tau)$ and $x(\tau) = X^{(k)}\psi_0(\tau)$. (4.3)

Integrating Eqn. (4.3) with E = $\frac{1}{2}$, we get

$X^{(k)} = \frac{1}{2}V^{(k)} + x(k-1)$ and

$x(k) = V^{(k)} + x(k-1).$ (4.4)

Therefore, $x(\tau) = \left(\frac{1}{2}V^{(k)} + x(k-1)\right)\psi_0(\tau)$

(4.5)

To solve Eqn. (4.2), we first substitute Eqn. (4.5) in $f(\tau, x(\tau), u(\tau))$. Then we express the resulting equation by STWS as

$$f\left(\tau, \left(\frac{1}{2}V^{(k)} + x(k-1)\right)\psi_0(\tau), u(\tau)\right) = F^{(k)}\psi_0(\tau)$$

(4.6)

Using Eqns. (4.2), (4.3), (4.5), and (4.6), we get

$mV^{(k)} = F^{(k)}.$ (4.7)

By solving Eqn. (4.7), the components of $V^{(k)}$ can be obtained. By substituting $V^{(k)}$ in Eqn. (4.4), we obtain block-pulse and discrete approximations of the state, respectively. For higher order non-linear system of IVPs, the system can be transformed into system of first order IVPs and then the STWS technique can be applied as mentioned above.

RK Methods Based on Variety of Means

The classical fourth order RK formula for solving IVPs of the form $\dot{x} = f(t, x)$ may be written as

$$x_{n+1} = x_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= x_n + \frac{h}{3}\sum_{i=1}^{3}\frac{k_i + k_{i+1}}{2}$$

$$= x_n + \frac{h}{3}\sum_{i=1}^{3}(AM)$$

where

$$k_1 = f(t_n, x_n)$$

$$k_2 = f\left(t_n + \frac{h}{2},\ x_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2},\ x_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(t_n + h,\ x_n + h k_3\right)$$

By replacing the AM by the various means such as HaM, CeM, and CoM, we get different RK formulae. The means HaM, CeM, and CoM are defined in terms of AM and Geometric Mean (GM). The formulae for various means are given in Table 1, in terms of two values x_1 and x_2. For $\dot{x} = f(t, x)$ the fourth order linear or non-linear methods using a variety of means which can be written in the form

$$x_{n+1} = x_n + \frac{h}{3}\left[\sum_{i=1}^{3}\text{Mean}\right]$$

where Mean includes AM, HaM, CeM and CoM which involve $k_i,\ 1 \le i \le 4$ as follows:

$$k_1 = f(t_n, x_n)$$
$$k_2 = f(t_n + a_1 h,\ x_n + a_1 h k_1)$$
$$k_3 = f(t_n + (a_2 + a_3)h,\ x_n + a_2 h k_1 + a_3 h k_2)$$
$$k_4 = f(t_n + (a_4 + a_5 + a_6)h,\ x_n + (a_4 h k_1 + a_5 h k_2 + a_6 h k_3))$$

and the parameters a_j, $1 \le j \le 6$ are shown in Table 2. The fourth order formulae, based on the Runge-Kutta scheme using various means, are given in Table 3.

Table 1 Formulae for Various Means

Mean	Notation	Formula	Formula in terms of AM & GM
Arithmetic Mean	AM	$\frac{x_1 + x_2}{2}$	--
Geometric Mean	GM	$\sqrt{x_1 x_2}$	--
Harmonic Mean	HaM	$\frac{2x_1x_2}{x_1 + x_2}$	$\frac{(GM)^2}{AM}$
Centroidal Mean	CeM	$\frac{2}{3}\left(\frac{x_1^2 + x_2^2 + x_1x_2}{x_1 + x_2}\right)$	$\frac{4(AM)^2 - (GM)^2}{3(AM)}$
Contraharmonic Mean	CoM	$\frac{x_1^2 + x_2^2}{x_1 + x_2}$	$\frac{2(AM)^2 - (GM)^2}{AM}$

Table 2 Values of the Parameters a_j

Parameters \ Means	AM	HaM	CeM	CoM
a_1	1/2	1/2	1/2	1/2
a_2	0	-1/8	1/24	1/8
a_3	1/2	5/8	11/24	3/8
a_4	0	-1/4	1/12	1/4
a_5	0	7/20	-25/132	-3/4
a_6	1	9/10	73/66	3/2

Table 3 Fourth Order RK Formulae

Mean	$x_{n+1} =$
AM	$x_n + \frac{h}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$
HaM	$x_n + \frac{2h}{3}\left[\frac{k_1 k_2}{k_1 + k_2} + \frac{k_2 k_3}{k_2 + k_3} + \frac{k_3 k_4}{k_3 + k_4}\right]$
CeM	$x_n + \frac{2h}{9}\left[\frac{k_1^2 + k_1k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2k_3 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_3k_4 + k_4^2}{k_3 + k_4}\right]$
CoM	$x_n + \frac{h}{3}\left[\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right]$

Numerical Examples

Example 1

Consider the stiff system of two non-linear differential equations given by

$$\dot{x}_1 = -1002x_1 + 1000x_2^2, \quad x_1(0) = 1,$$

$$\dot{x}_2 = x_1 - x_2(1 + x_2), \qquad x_2(0) = 1, \quad (6.1)$$

As the independent variable't' does not appear explicitly in Eqn. (6.1), it is an autonomous system. The exact solution of this system is given by

$x_1(t) = e^{-2t}$ and $x_2(t) = e^{-t}$.

For this problem, the Jacobian at t = 0 is

$J = \begin{bmatrix} -1002 & 2000 \\ 1 & -3 \end{bmatrix}$, whose eigenvalues are

λ = [-1, -1004] and Stiffness Ratio SR = 1004. Hence it is classified as stiff at t = 0. Further, this problem is super stable since there is at least one eigenvalue with a large negative real part.

The discrete solutions obtained by using STWS technique with m = 300 and RK methods based on variety of means with h = 0.001 are compared with the corresponding exact solution. The results are shown in Tables 4 and 5. The error graphs of these methods are shown in Fig. 1 and Fig. 2.

Example 2

Consider the following stiff non-linear system:

$$\dot{x}_1 = -40.2x_1 + 19.6x_2 + \frac{\cos(t)}{1+t} - \frac{\sin(t)}{(1+t)^2} + \frac{40.2\sin(t)}{1+t},$$

$$\dot{x}_2 = 19.6x_1 - 10.8x_2 - \frac{19.6\sin(t)}{1+t},$$

with the initial conditions

$$x_1(0) = 3, \qquad x_2(0) = 1.$$

The exact solution of this system is given by

$$x_1 = e^{-t} + 2e^{-50t} + \frac{\sin(t)}{1+t},$$

$$x_2 = 2e^{-t} - e^{-50t}.$$

For this problem, the Jacobian at t = 0 is

$J = \begin{bmatrix} -40.2 & 19.6 \\ 19.6 & -10.8 \end{bmatrix}$ whose eigen values are

λ = [-50, -1] and stiffness ratio SR = 50. Hence it is classified as stiff at t = 0 and is also super stable.

The discrete solutions obtained by using STWS technique and RK methods based on variety of means are compared with the corresponding exact solution. The results are shown in Tables 6 and 7. The error graphs of these methods are shown in Fig. 3 and Fig. 4.

Example 3

Consider the following second order stiff non linear equation:

$$\ddot{x} = \frac{1}{x^3} - \left\{\frac{(1+t)^6 + 2t^3}{t^4(1+t)^2}\right\}x, \quad x(1) = 0.5,$$

$$\dot{x}(1) = 0.25.$$

The exact solution is $x = \frac{t}{1+t}$.

By putting $x = x_1$ and $\dot{x} = x_2$, the above system can be transformed into system of two first order non-linear equations as follows:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \frac{1}{x_1^3} - \left\{\frac{(1+t)^6 + 2t^3}{t^4(1+t)^2}\right\}x_1,$$

with the initial conditions $x_1(1) = 0.5$ and

$x_2(1) = 0.25.$

Exact solution of this transformed system is given by

$$x_1 = \frac{t}{1+t} \text{ and } x_2 = \frac{1}{(1+t)^2}.$$

The discrete solutions obtained by using STWS technique and RK methods based on variety of means are compared with the corresponding exact solution. The results are shown in Tables 8 and 9. The error graphs of these methods are shown in Fig. 5 and Fig. 6.

Example 4

Consider nuclear reactor model characterized by stiff non-linear system that arises in nuclear reactor theory

$$\dot{x}_1 = 0.01 - (0.01 + x_1 + x_2)\left(1 + (x_1 + 1000)(x_1 + 1)\right),$$

$$\dot{x}_2 = 0.01 - (0.01 + x_1 + x_2)\left(1 + x_2^2\right),$$

with the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 0.$$

For this problem, the Jacobian at t = 0 is $J = \begin{bmatrix} -1011.01 & -1001 \\ -1 & -1 \end{bmatrix}$ whose eigen values are λ = [-1012, -0.0098913] and stiffness ratio SR = 102312. This indicates that this problem is very stiff at t = 0 and it is also super stable. The discrete solutions are obtained using STWS technique and RK methods based on variety of means. The results are shown in Tables 10 and 11.

Example 1

Table 4 Absolute Error in x_1 (t)

Time t	STWS	RK Methods Based on AM	HaM	CeM	CoM
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2	1.17E-07	1.36E-06	1.36E-06	1.36E-06	1.36E-06
4	3.18E-09	7.17E-08	7.17E-08	7.17E-08	7.17E-08
6	5.12E-11	3.09E-09	3.10E-09	3.10E-09	3.10E-09
8	2.67E-12	8.92E-11	8.92E-11	8.92E-11	8.92E-11
10	5.86E-13	1.69E-12	1.69E-12	1.69E-12	1.69E-12

Table 5 Absolute Error in x_2 (t)

Time t	STWS	RK Methods Based on AM	HaM	CeM	CoM
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2	2.51E-07	5.04E-06	5.04E-06	5.02E-06	5.02E-06
4	6.79E-08	1.96E-06	1.96E-06	1.96E-06	1.96E-06
6	1.38E-08	6.24E-07	6.24E-07	6.25E-07	6.25E-07
8	2.49E-09	1.33E-07	1.33E-07	1.33E-07	1.33E-07
10	4.21E-10	1.86E-08	1.86E-08	1.86E-08	1.86E-08

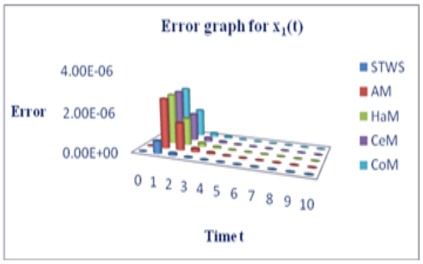

Fig. 1

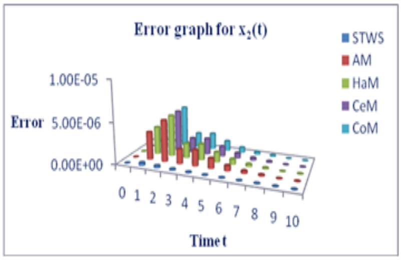

Fig. 2

Example 2

Table 6 Absolute Error in $x_1(t)$

Time		RK Methods Based on			
t	STWS	AM	HaM	CeM	CoM
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
5	1.32E-07	1.97E-06	1.88E-06	1.97E-06	2.00E-06
10	2.50E-08	6.52E-06	6.52E-06	6.52E-06	6.52E-06
15	4.60E-08	7.67E-07	7.71E-07	7.67E-07	7.64E-07
20	2.52E-08	3.80E-06	3.80E-06	3.80E-06	3.80E-06
25	2.44E-08	2.23E-06	2.23E-06	2.23E-06	2.23E-06
30	2.66E-08	1.44E-06	1.44E-06	1.44E-06	1.44E-06

Table 7 Absolute Error in x_2 (t)

Time		RK Methods Based on			
t	STWS	AM	HaM	CeM	CoM
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
5	3.49E-08	4.17E-06	3.97E-06	4.18E-06	4.24E-06
10	1.43E-08	1.18E-05	1.18E-05	1.18E-05	1.18E-05
15	3.64E-09	6.98E-07	7.01E-07	6.99E-07	6.95E-07
20	4.34E-09	7.21E-06	7.21E-06	7.22E-06	7.22E-06
25	1.96E-09	3.62E-06	3.61E-06	3.61E-06	3.61E-06
30	4.09E-09	2.99E-06	2.99E-06	2.99E-06	2.99E-06

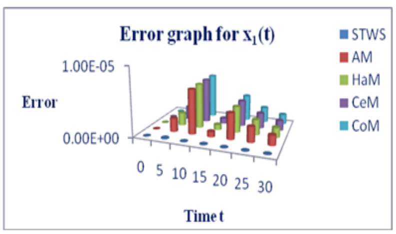

Fig. 3

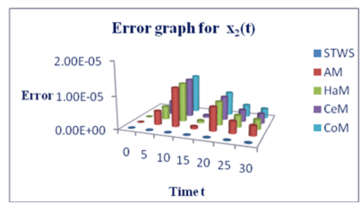

Fig. 4

Example 3

Table 8 Absolute Error in x_1 (t)

Time		RK Methods Based on			
t	STWS	AM	HaM	CeM	CoM
1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
5	6.40E-07	1.19E-06	1.13E-06	1.13E-06	1.07E-06
10	5.11E-07	2.15E-06	2.21E-06	2.03E-06	2.09E-06
15	4.36E-07	1.61E-06	1.19E-06	1.07E-06	1.19E-06
20	1.89E-07	1.61E-06	2.15E-06	1.79E-06	1.79E-06
25	4.53E-07	6.56E-07	1.79E-07	5.96E-08	1.19E-07
30	3.49E-07	3.93E-06	4.59E-06	4.35E-06	4.41E-06

Table 9 Absolute Error in x_2 (t)

Time		RK Methods Based on			
t	STWS	AM	HaM	CeM	CoM
1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
5	8.13E-08	3.45E-07	9.44E-07	3.13E-07	1.10E-07
10	3.51E-07	3.52E-06	4.71E-06	4.62E-06	3.98E-06
15	6.26E-07	5.15E-06	6.41E-06	5.63E-06	5.62E-06
20	1.04E-06	8.05E-07	1.51E-06	2.12E-06	1.69E-06
25	5.14E-07	2.12E-06	3.06E-06	2.28E-06	2.42E-06
30	8.22E-07	1.29E-06	1.90E-07	2.55E-07	1.17E-07

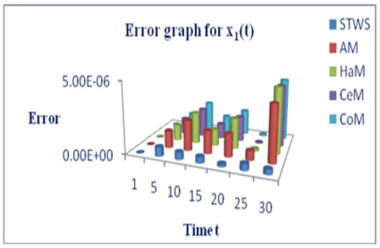

Fig. 5

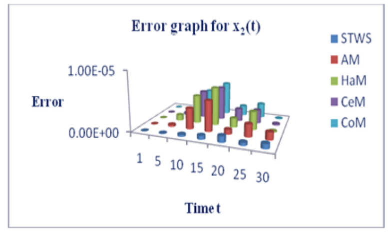

Fig. 6

Example 4

Table 10 Discrete Solution of x_1 (t)

Time		RK Methods Based on			
t	STWS	AM	HaM	CeM	CoM
0.0	0	0	0	0	0
0.2	-0.0119657	-0.011965755	-0.011962901	-0.011966682	-0.011968782
0.4	-0.0139617	-0.013961677	-0.013958823	-0.013962604	-0.013964703
0.6	-0.0159576	-0.015957553	-0.015954699	-0.015958481	-0.01596058
0.8	-0.0179535	-0.017953437	-0.017950581	-0.017954364	-0.017956464
1.0	-0.0199493	-0.019949406	-0.019946551	-0.019950334	-0.019952433
1.2	-0.0219452	-0.021945374	-0.021942521	-0.021946302	-0.021948401
1.4	-0.0239411	-0.023941174	-0.023938319	-0.023942102	-0.023944201
1.6	-0.025937	-0.025936987	-0.025934132	-0.025937915	-0.025940014
1.8	-0.0279328	-0.027932955	-0.027930100	-0.027933883	-0.027935982
2.0	-0.0299287	-0.029928925	-0.029926069	-0.029929852	-0.029931951

Table 11 Discrete Solution of x_2 (t)

Time		RK Methods Based on			
t	STWS	AM	HaM	CeM	CoM
0.0	0	0	0	0	0
0.2	0.0019859	0.001985957	0.001983103	0.001986884	0.001988984
0.4	0.0039819	0.003981919	0.003979065	0.003982846	0.003984946
0.6	0.0059778	0.005977836	0.005974982	0.005978764	0.005980863
0.8	0.0079738	0.007973761	0.007970907	0.007974689	0.007976788
1.0	0.0099697	0.009969772	0.009966917	0.009970699	0.009972799
1.2	0.0119656	0.011965782	0.011962928	0.01196671	0.011968809
1.4	0.0139615	0.013961623	0.013958768	0.013962551	0.01396465
1.6	0.0159575	0.015957478	0.015954623	0.015958406	0.015960505
1.8	0.0179533	0.017953489	0.017950634	0.017954417	0.017956516
2.0	0.0199492	0.0199495	0.019946644	0.019950427	0.019952526

A critical study on errors in stiff non-linear systems

In Examples 1 – 3, the absolute errors between the discrete solutions and the exact solutions have been determined and presented in Tables 4 - 9. For a better analysis of the system given in Example 4 which is a stiff non-linear autonomous system with high stiffness ratio whose analytic solution is not known, and to demonstrate the applicability of the STWS technique, a critical study on the absolute errors of the STWS solutions has been carried as explained below:

- First, the STWS solutions are obtained in the solution space [0, 2] for different values of 'm', for example, m = 900, 1000, 1100, 1200, 1300, 1400.
- The absolute errors between the STWS solutions obtained at two consecutive values of 'm', say m_1 and m_2, have been determined which may be denoted as $Err_{m_1}^{m_2}$.

All these details have been presented in Tables 12 and 13 for x_1 and x_2 respectively. From these two tables, the maximum errors obtained for x_1 and x_2 in [0, 2] have been identified and presented in Table-14.

Table 12

Time t	Error in x_1				
	Err_{900}^{1000}	Err_{1000}^{1100}	Err_{1100}^{1200}	Err_{1200}^{1300}	Err_{1300}^{1400}
0	0	0	0	0	0
0.25	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07
0.50	2.431E-06	1.888E-06	1.492E-06	1.198E-06	9.73E-07
0.75	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07
1.00	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07
1.25	2.431E-06	1.888E-06	1.492E-06	1.198E-06	9.73E-07
1.50	2.431E-06	1.888E-06	1.492E-06	1.198E-06	9.73E-07
1.75	2.432E-06	1.887E-06	1.492E-06	1.198E-06	9.73E-07
2.00	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07

Table 13

Time t	Error in x_2				
	Err_{900}^{1000}	Err_{1000}^{1100}	Err_{1100}^{1200}	Err_{1200}^{1300}	Err_{1300}^{1400}
0	0	0	0	0	0
0.25	2.432E-06	1.888E-06	1.492E-06	1.198E-06	9.73E-07
0.50	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07
0.75	2.432E-06	1.887E-06	1.493E-06	1.197E-06	9.74E-07
1.00	2.431E-06	1.888E-06	1.492E-06	1.198E-06	9.73E-07
1.25	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07
1.50	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07
1.75	2.432E-06	1.887E-06	1.493E-06	1.197E-06	9.74E-07
2.00	2.432E-06	1.888E-06	1.492E-06	1.197E-06	9.74E-07

Table 14

Variables	Maximum Error in [0, 2]				
	Err_{100}^{200}	Err_{200}^{300}	Err_{300}^{400}	Err_{400}^{500}	Err_{500}^{600}
x_1	2.432E-06	1.888E-06	1.492E-06	1.198E-06	9.740E-07
x_2	2.432E-06	1.888E-06	1.492E-06	1.198E-06	9.740E-07

Conclusion

In this paper, the applicability and effectiveness of the STWS technique in determining discrete solutions for the stiff non-linear systems has been studied by comparing with the discrete solutions obtained using RK methods based on AM, HaM, CeM and CoM. To demonstrate the effectiveness of the STWS technique, four examples of stiff non-linear systems have been considered. The stiffness ratios have been determined for the systems under discussion.

From the Tables 4 - 9, it is evident that the absolute errors in STWS solutions of the stiff non-linear systems given in Examples 1 – 3 are lesser than the absolute errors in the solutions of the RK methods by variety of means. This proves that the STWS technique has an edge over RK methods based on variety of means in terms of accuracy.

From the Tables 10 and 11, it is observed that the STWS solutions of the stiff non-linear system given in Examples 4 (whose analytic solution is not known) agree well with that of the RK methods based on variety of means. Further, Table 14 shows that the errors in x(t) decreases as the value of 'm' increases.

Hence, in general, it is concluded that the proposed STWS technique is highly stable and very much applicable for solving stiff linear and non-linear autonomous as well as non-autonomous problems including highly stiff problems.

References

[1] J. R. Cash, On the Integration of Stiff System of ODEs Using Extended Backward Differentiation Formulae, *Numer. Math.* 34 (1980), pp. 235 – 246.

[2] L. Lopez, An Explicit Two-step method for Solving Stiff Systems of Ordinary Differential Equations, *Int. J. Comput. Math.* 22 (1987), pp. 271 - 285.

[3] G. A. F. Ismail, and I. H. Ibrahim, A New Higher Order Effective P-C Methods for Stiff Systems, Mathematics *and Computers in Simulation* 47 (1998), pp. 541 – 552.

[4] C. H. Hsiao, Haar Wavelet Approach to Linear Stiff Systems, *Mathematics and Computers in Simulation* 64 (2004), pp. 561 – 567.

[5] N. M. Bujurke, C. S. Salimath and S. C. Shiralashetti , Numerical Solution of Stiff Systems from Nonlinear Dynamics using single-term Haar wavelet series, *Nonlinear Dyn.* 51 (2008), pp. 595–605.

[6] G.P. Rao, K.R. Palanisamy, and T. Srinivasan, Extension of computation beyond the limit of normal interval in Walsh Series analysis of dynamical systems, *IEEE Trans. Autom. Control 25* (1980), pp. 317 – 319.

[7] K.R. Palanisamy, Analysis and optimal control of linear systems via single-term Walsh series approach, *Int. J. Systems Sci.* 12 (1981), pp. 443 – 454.

[8] R. Subbayyan, K. Muhammad Zakariah and K. R. Palanisamy, Optimal Control of Singularly Perturbed Linear Systems: Single Term Walsh Series Approach, *Int. J. Systems Sci.* 13 (1982), pp. 1339 – 1343.

[9] K. R. Palanisamy and V. P. Arunachalam, Analysis of Bilinear Systems via Single-Term Walsh Series, *Int. J. Control* 41 (1985), pp. 541 – 547.

[10] K. Balachandran and K. Murugesan, Analysis of non-linear singular systems via STWS method, *Int. J. Comp. Math.* 36 (1990), pp. 9 – 12.

[11] K. Murugesan, D. Paul Dhayabaran, and David J. Evans, Analysis of second order multivariable linear system using single-term Walsh series technique and Runge Kutta method, *Int. J. Comp. Math.* 72 (1999), pp. 367 – 374.

[12] B. Sepehrian and M. Razzaghi, Solution of Time-varying Singular Nonlinear Systems by Single Term Walsh Series, *Mathematical Problems in Engineering* 2003:3 (2003), pp. 129 – 136.

[13] A. Emimal Kanaga Pushpam, D. Paul Dhayabaran and E. C. Henry Amirtharaj, Numerical solution of higher order systems of IVPs using generalized STWS technique, *Applied Mathematics and Computation* 180 (2006), pp. 200 – 205.

[14] D. Paul Dhayabaran, A. Emimal Kanaga Pushpam and E. C. Henry Amirtharaj, Generalized STWS Technique for Higher Order Time-varying Singular Systems, *Int. J. Comput. Math*., 84 (2007), pp. 395 – 402.

[15] R. Yaakub and D. J. Evans, New Runge-Kutta Starters for Multi-step Methods, *Int. J. Comput. Math.* 71 (1999), pp. 99 – 104.

[16] Abdul-Majid Wazwaz, A comparison of modified Runge-Kutta formulas based on a variety of means, *Int. J. Comput. Math*. 50 (1994), pp. 105 – 112.

[17] B. B. Sannugi and D. J. Evans, A new fourth order Runge-Kutta formula based on the Harmonic Mean, *Int. J. Comput. Math.* 50 (1999), pp. 113 – 118.

[18] J. D. Lambert, Numerical Methods for Ordinary Differential Systems: The Initial Value Problem, John Wiley & Sons, Toronto, 1991.

[19] L. F. Shampine and C. W. Gear, A User's View of Solving Stiff Ordinary Differential Equations, *SIAM Rev.* 21 (1979), pp.1-17.