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Operator Duals of Vector Sequence Spaces

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Abstract

T. Balasubramaniam and A. Pandiarani [1] have defined the sequence spaces $\Gamma(x)$. λ (x), G(x) and have studied some topological property of their Köthe-Toeplitz duals in operator form. In this paper we define the new sequence spaces $\Gamma_0(x, p)$, $\Gamma_e(x, p)$ and $\Gamma_\infty(x,p)$ and have studied some topological property of their Köthe-Toeplitz duals in operator form.

Key Words: Analytic sequences, Entire sequences, Köthe-Toeplitz duals

Introduction

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [6]. In fact, the basic problems in the theory of complex sequence space or scalar valued sequence space deal with the transformation of complex sequence by infinite matrices of complex numbers. The basic results in this regard may be seen in the books of Cooke [3] ,Hardy [4] and Maddox [5].

When the infinite matrices of complex numbers operate on a complex sequence, we come across an infinite series whose convergence has given rise to the concept of β-duals also called as Köthe-Toeplitz duals. Thus if E is a set of complex sequences, then β-dual of E is denoted by E^β and is defined as

$$E^\beta = \{a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x = (x_k) \in E, \text{ where } \omega \text{ is the set of all complex sequences}\}$$

The main result of Köthe and Toeplitz [2] is concerned with α-duals. If ω is the set of all complex sequence spaces, then the α-dual of E, (where $E \subset \omega$) is denoted by E^α and is defined as

$$E^\alpha = \{a = (a_k) \in \omega : \sum_{k=1}^{\infty} | a_k x_k | \text{ converges for all } x = (x_k) \in E\}$$

An account of the theory of α-duals in the scalar case may be found in G. Köthe [6]. Another dual, the γ-dual of the set $E \subset w$ is denoted by E^γ and is defined as

$$E^\gamma = \{a = (a_k) \in \omega : \sup_n \left| \sum_{k=1}^{n} a_k x_k \right| < \infty \text{ for all } x = < x_k > \in E\}$$

After Robinson's paper in 1995 [7] where in he considered the action of matrices operators on Banach space valued sequence, a decisive break occurred. This gave the concept of generalized Köthe-Toeplitz duals as given below.

Let X and Y be a Banach spaces and E((X) is a set of X-valued sequences i.e. E(x) is a non-empty set of sequence $x = (x_k)$ with $x_k \in X$. The several generalized Köthe-Toeplitz duals arise when the element complex sequence (a_k) of α-duals and β-duals set is replaced by a sequence (A_k) of linear operators, where each A_k is a linear operator from X into Y. Thus, we define β-dual of E(X) as

$$E^\beta(x) = \{(A_k) : \sum_{k=1}^{\infty} (A_k x_k) \text{ converges in Y, for each } (x_k) \in E(X)\}$$

and α-dual of E (X) as

$$E^\alpha(X) = \{(A_k) : \sum_{k=1}^{\infty} \| A_k x_k \| \text{ converges for all } (x_k) \in E\}$$

Furthermore, Maddox [8] has also given the convergence of sequence of operators which are not necessarily bounded.

Notation and Terminology

We denote the set of all natural, real and complex numbers by N, R and C respectively. A sequence $x = (x_k)$ is said to be an entire sequence if $\lim_{k\to\infty} | x_k |^{\frac{1}{k}} = 0$ and a sequence x = (x_k) is said to be an analytic sequence if $\left\{ x_k \,|^{\frac{1}{k}} \right\}$ is bounded.

If (X, ||.||) is any Banach space over C, then we define

$$\overline{|_0(x,p)} = \{\bar{x} = < x_k >: \in X, \underset{k\to\infty}{Limit} \| x_k \|^{\frac{1}{p_k}} = 0\}$$

$$\overline{|_c(x,p)} = \{\bar{x} = < x_k >: \in X, \underset{k\to\infty}{Limit} \| x_k \|^{\frac{1}{p_k}} = l\}$$

$$\overline{|_\infty(x,p)} = \{\bar{x} = < x_k >: \in X, \underset{k\to\infty}{Limit} \| x_k \|^{\frac{1}{p_k}} = \infty\}$$

where < p_k > is a sequence of positive real number such that $p_k \geq k$, for each k and *l* is a non-negative real number.

Let X and Y are Banach spaces and < A_k > is a linear operator from X onto Y and E is non-empty set of sequence x = (x_k) with $x_k \in X$.

Then, we define -duals, -duals (m,) duals and (m,) duals of E as

E^α = {< A_k > : $\sum_{k=1}^{\infty} \| A_k x_k \|$ converges for all x = < x_k > $\in$E}

E^β = {< A_k > : $\sum_{k=1}^{\infty} A_k x_k$ converges is Y for all x = < x_k > $\in$E}

$E^{m,\alpha}$ = {< A_k > : < A_k > is a sequence of linear operator but not necessarily bounded such that for some m; $\sum_{k=m}^{k=\infty} \| A_k x_k \|$ converges for all < x_k > $\in$ E}

$E^{m,\beta}$ = {< A_k > < A_k > is a sequence of linear operator but not necessarily bounded such that for some m; $\sum_{k=m}^{k=\infty} A_k x_k$ converges in Y for all < x_k > $\in$ E}

Let B (X, Y) denote the Banach spaces of the bounded linear operators from X into Y with the usual operator norm.

From the definition of the spaces, $\overline{|_0(X,p)}, \overline{|_c(X,p)}, \overline{|_\infty(X,p)}$. . It is obvious that

$$\overline{|_0(X,p)} \subset \overline{|_c(X,p)} \subset \overline{|_\infty(X,p)}.$$

Thus, if T $\in$ B(X, Y), the operator norm of T is

|| T || = sup {|| T (x) ||}: x$\in$S}, where

S = {x $\in$ X: || x || $\leq$ 1} is a closed unit sphere in X.

Definitions: Let < T_k > = < T_1, T_2, T_3, ...> be a sequence in B (X, Y). Then, the group norm of (T_k) is

$$\| (T_k) \| = \sup \left\| \sum_{k=1}^{n} T_k(x_k) \right\|$$

where supremum is over all n $\in$ N and all $x_k \in$ S.

Main Results

Lemma 1. If (T_k) is a sequence in B (X, Y) and we write R_n = (T_n, T_{n+1},..), then

(i) || T_m || $\leq$ || R_n ||, for all m $\geq$ n

(ii) || R_{n+1} || $\leq$ || R_n ||, for all n $\in$ N

(iii) $\left\| \sum_{k=n}^{n+p} T_k x_k \right\| \leq \| R_n \|$ .max {|| x_k || : n $\leq$ k $\leq$ n + p}

for any x_k, for all n $\in$ N and all non-negative integer p.

Lemma 2. If (T_k) is a sequence in B (X, Y), then

$$\Sigma \| T_k \| < \infty \quad \Rightarrow \sup_{k\geq 1} \| T_k \| < \infty$$

Theorem 1. < A_k > $\in$ $\overline{|_0(X,p)}^{m,\beta}$ iff $\exists$ m $\in$ N such that

(i) $A_k \in$ B (X, Y) ; for each k $\geq$ m.

(ii) $\sup_{k\geq m} \| A_k \|^{\frac{1}{p_k}} < \infty$

Proof : Suppose the condition (i) and (ii) holds good.

Let < x_k > $\in \overline{|_0(X,p)} \Rightarrow \lim_{k\to\infty} = 0$

$\Rightarrow$ To each $\in$ > 0 $\exists$ +ve integer m such that

$\| x_k \|^{1/p_k} < \in$ for all k $\geq$ m ...(3.1.1)

Since, $\sup_{k\geq m} \| A_k \|^{1/p_k} < \infty$

$\Rightarrow$ $\exists$ a real number M > 0 such that M $\in$ < 1

and $\| A_k \|^{\frac{1}{p_k}}$ < M for all k $\geq$ m ...(3.1.2)

from (1) and (2), $\| A_k \|^{1/p_k} \| x_k \|^{1/p_k}$ < M$\in$

$\Rightarrow$ $(\| A_k \| \, \| x_k \|)^{1/p_k}$ < M$\in$

$\Rightarrow$ || A_k || . || x_k || < $(M\in)^{p_k}$

(since M $\in$ < 1 and $p_k \geq$ k $\Rightarrow$ $(M\in)^{p_k} \leq (M\in)^k$).

$\Rightarrow$ || A_k || . || x_k || < $(M\in)^k$ ...(3.1.3)

By result, || A_k . (x_k) || || A_k || . || x_k ||, k $\geq$ m. ...(3.1.4)

From (3.1.3) and (3.1.4), || A_k . (x_k) || < $(M\in)^k$ for all k $\geq$ m.

$$\left\| \sum_{k=m}^{\infty} A_k x_k \right\| \leq \sum_{k=m}^{\infty} \| A_k x_k \| < \sum_{k=m}^{\infty} (M \in)^k$$

$$= \quad (M\in)^m \left[\frac{1}{1 - M\in} \right] = \frac{(M\in)^m}{1 - M\in} < \infty$$

$\Rightarrow$ $\sum_{k=1}^{\infty} A_k x_k$ converges in Y for all < x_k > $\overline{|_0(X,p)}$

consequently,

< A_k > $\in$ $\overline{|_0(X,p)}^{m,\beta}$.

Conversely, Let < A_k > $\in$ $\overline{|_0(X,p)}^{m,\beta}$

But that no m exists for which $A_k \in$ B (X, Y) for all k $\geq$ m.

$\Rightarrow$ $\exists$ a sequence < k_i > of natural no. such that $k_i \geq$ i and $A_{k_i} \notin$ B (x, y), for each i

$\Rightarrow$ || A_{k_i} || = ∞ where $k_i \geq$ i, for each $k_i \geq$ i

$\Rightarrow$ || A_{k_i} || = sup { || A_{k_i} (z) || : for each z $\in$ X and || z || $\leq$ 1} = ∞

$\Rightarrow \sup_{z\in S} \| A_{k_i}(z) \| = \infty$ where $k_i \geq i$, for each $k_i \geq i$

$\Rightarrow \exists$ a sequence $< z_i >$ in S such that

$\| A_{k_i}(z_i) \| > i^{2p_{k_i}}$ where $k_i \geq i$, for each $k_i \geq i$

Define $x_k = \begin{cases} 0, & k \neq k_i \\ \dfrac{z_i}{i^{2p_{k_i}}} & k = k_i \end{cases}$

For $k \neq k_i$, $\| x_k \|^{1/p_k} = 0$

and for $k = k_i$, $\| x_k \|^{1/p_k} = \| x_{k_i} \|^{1/p_k} = \| z_i \|^{1/p_k} / i^2 \leq 1/i^2 \to 0$ as $i \to \infty$, i.e. $k_i \to \infty$, i.e. $k \to \infty$ (since $\| z_i \| \leq 1$)

$\Rightarrow \| x_k \|^{1/p_k} \to 0$ as $k \to \infty$.

Hence, $< x_k > \in \overline{l_0 (X, p)}$

For $k = k_i$, $\| A_k x_k \| = \| A_{k_i}(x_{k_i}) \|$

$$= \left\| A_{ki}\left(\frac{z_i}{i^{2p_{k_i}}}\right)\right\| = \frac{\| A_{ki}(z_i) \|}{i^{2p_{k_i}}} > 1 \text{ for large}$$

value of k.

Hence, $\sum_{k=m}^{\infty} A_k(x_k)$ does not converge, which is contradiction to our assumption. Hence our assumption is wrong.

Therefore, $\exists$ m$\in$ N such that $A_k \in$ B (X, Y), for $k \geq m$.

Again suppose if possible (ii) fails

$\Rightarrow$ There does not exist any $m \in N$ such that $\sup_{k \geq m} \| A_k \|^{\frac{1}{p_k}} < \infty$

$\Rightarrow$ To each $m \in N$, $\sup_{k\geq m} \| A_k \|^{\frac{1}{p_k}} = \infty$

$\Rightarrow \sup_{k\geq 1}\{\| A_k \|\}^{\frac{1}{p_k}} = \infty$

$\Rightarrow \exists$ a sequence $< k_i >$ of natural no. such that $\| A_{k_i} \|^{1/p_{k_i}} > 2i$ for $i \in N$.

Again since, $(1/2^k)\| A_k \| \leq \| A_k \|$ and $\| A_k \| = \sup \{\| A_k(z) \|$: for each $z \in S\}$

$\Rightarrow (1/2^k) \| A_k \| \leq \sup \{\| A_k(z) \|$: for each $z \in S \}$

$\Rightarrow \exists$ a sequence $< z_k >$ in S such that $(1/2^k) \| A_k \| \leq \| A_k (z_k) \|$

$\Rightarrow \| A_k \| \leq 2^k \| A_k . (z_k) \|$ i.e. $\| A_k (z_k) \| \geq \| A_k \| / 2^k$

Define $x_k = Z_k / i^{p_k}$ for $k = k_i$ and $x_k = 0$ for $k \neq k_i$;

For $k = k_i$

$$\| x_k \|^{\frac{1}{p_k}} = \left\| \frac{z_k}{i^{pk}} \right\|^{\frac{1}{p_k}} = \frac{1}{i}.\| z_k \|^{\frac{1}{p_k}} \to 0 \text{ as } i \to \infty, \text{ i.e. } k_i = k \to \infty$$

and for $k \neq k_i$, $\| x_k \|^{1/p_k} = 0$

Then $< x_k > \in \overline{l_0 (X, p)}$

But for $k = k_i$,

$$\| A_k x_k \| = \left\| A_k\left(\frac{z_k}{i^{p_k}}\right)\right\|$$

$$= \frac{1}{i^{p_k}} \| A_k(z_k) \| \geq \frac{\| A_k \|}{2^k . i^{pk}} \geq \frac{\| A_k \|}{(2i)^{p_k}} > 1 \text{ for large}$$

value of k.

$\Rightarrow \Sigma A_k (x_k)$ diverges, which is contradiction to our assumption. Hence our assumption is wrong.

Therefore, $\exists$ $m \in N$ such that $\sup_{k\geq m} \| A_k \|^{\frac{1}{p_k}} < \infty$

Theorem : 2. $< A_k > \in \overline{l_0 (X, p)}^{(m,\alpha)}$ iff $\exists$ $m \in N$ such that (i) $A_k \in$ B (X, Y), for $k \geq m$.

and (ii) $\left(\sum_{k=1}^{\infty} \| A_k \|\right) < \infty$

Proof : Suppose the condition (i) and (ii) holds good.

Let $x = (x_k) \in \overline{l_0 (X, p)} \Rightarrow \underset{k\to\infty}{Limit} \| x_k \|^{\frac{1}{p_k}} = 0$

$\Rightarrow$ for given $\in > 0$ $\exists$ a +ve integer m such that $\| x_k \|^{1/p_k} < \in$ for all $k \geq m$.

i.e. $\| x_k \| < \in^{p_k}$, for all $k \geq m$ ...(3.2.1)

since $\left(\sum_{k=m}^{\infty} \| A_k \|\right) < \infty$

$\Rightarrow \exists$ real number $M > 0$ such that $M \in < 1$

and $\sum_{k=m}^{\infty} \| A_k \| \leq M$, for all $k \geq m$.

$\Rightarrow \| A_k \| < M \leq (M^{p_k})$, for all $k \geq m$. ...(3.2.2)

from (3.2.1) and (3.2.2), $\| A_k \| . \| x_k \| < M^{p_k} . \in M^{p_k} = (M\in)^{p_k} \leq (M\in)^k$

$$\sum_{k=m}^{\infty} \| A_k x_k \| < \sum_{k=m}^{\infty} (M \in)^k = (M \in)^m \left[\frac{1}{1-M \in}\right] < \infty$$

(since $M \in < 1$ and $p_k \geq k \Rightarrow (M\in)^{p_k} \leq (M\in)^k$).

$\Rightarrow \sum_{k=m}^{\infty} \| A_k x_k \|$ converges.

consequently, $< A_k > \in \overline{l_0 (X, P)}^{(m,\alpha)}$

conversely, suppose $< A_k > \in \overline{l_0 (X, P)}^{(m,\alpha)}$

and there exist no m such that $A_k \in$ B (X, Y) for $k \geq m$

$\Rightarrow \exists$ a sequence $< k_i >$ of natural numbers such that $A_{k_i} \notin$ B (X, Y), for each i

$\Rightarrow \| A_{k_i} \| = \infty$, for each i

$\Rightarrow \| A_{k_i} \| = \sup \{\| A_{k_i}(z) \|$: for each $z \in X$ and $\| z \| \leq 1\} = \infty$

$\Rightarrow \sup_{\forall z\in S} \| A_{k_i}(z) \| = \infty$, for each $z \in S$

i.e. $\| z \| \leq 1$

$\Rightarrow \exists$ a sequence $< z_i >$ in S such that $\| A_{k_i}(z_i) \| > i^{2p_{k_i}}$ where $k_i \geq i$

Define $x_k = Z_i / i^{2\,p_k}$ for $k = k_i$ and $x_k = 0$ for $k \neq k_i$;

For $k = k_i$

$\| x_k \|^{1/p_k} = \| Z_i / i^{2\,p_k} \|^{1/p_k} = \| z_i \|^{1/p_k} / i^2 \to 0$ as $i \to \infty$, i.e. $k = k_i \to \infty$

and for $k \neq k_i$, $\| x_k \|^{1/p_k} = 0$

Then $< x_k > \overline{l_0\,(X, P)}$

for $k = k_i \geq m$,

$\| A_k (x_k) \| = \| A_k (Z_i / i^{2\,p_k}) \| = \| A_k (Z_i) \| \, (1 / i^{2\,p_k}) > 1$ for every value of k.

$\Rightarrow \sum_{k=m}^{\infty} \| A_k x_k \|$ does not converge, which is contradiction.

Hence, our assumption is wrong.

Therefore, $\exists\, m \in N$ such that $A_k \in B(X, Y)$, for each $k \geq m$.

Again suppose (ii) fails

i.e. $\left(\sum_{k=m}^{\infty} \| A_k \| \right) = \infty$

Then $\exists$ a strictly increasing sequence $< n_i >$ such that $\sum_{k=1+n_i}^{k=n_{i+1}} \| A_k \| > (2i)^{p_k}$

and $\exists$ a sequence $< z_k >$ in S such that $2^k \| A_k (z_k) \| \geq \| A_k \|$

Define $x_k = Z_k / i^{p_k}$ for $n_i < k \leq n_{(i+1)}$ and $x_k = 0$ otherwise ;

For $k = k_i$

$$\| x_k \|^{\frac{1}{p_k}} = \left\| \frac{z_k}{i^{p_k}} \right\|^{\frac{1}{p_k}} = \frac{\| z_k \|^{\frac{1}{p_k}}}{i} \to 0 \text{ as } I \to \infty \text{ i.e. } k = ki \to \infty$$

For $k \neq k_i$, $\| x_k \|^{1/p_k} = 0$

Then $< x_k > \in \overline{l_0\,(X, P)}$

But

$$\sum_{k=1+n_i}^{n_{i+1}} \| A_k x_k \| = \sum_{k=1+n_i}^{n_{i+1}} \left\| A_k \left(\frac{z_k}{i^{p_k}} \right) \right\| = \sum_{k=1+n_i}^{n_{i+1}} \frac{\| A_k (Z_k) \|}{i^{p_k}}$$

$$\geq \sum_{k=1+n}^{n_{i+1}} \frac{\| A_k \|}{2^k , i^{p_k}} \geq \sum_{k=1+n}^{n_{i+1}} \frac{\| A_k \|}{(2i)^{p_k}} > 1$$

$\Rightarrow \sum_{k=m}^{\infty} \| A_k x_k \|$ does not converge, which is a contradiction.

Hence our assumption is wrong

Therefore, $\exists\, m \in N$ such that

$$\sum_{k=m}^{\infty} \| A_k \| < \infty.$$

Theorem : 3. $< A_k > \in \overline{l_\infty\,(X, p)}^{(m,\beta)}$ iff

(i) $< A_k > \in \overline{l_0\,(X, p)}^{(m,\beta)}$

(ii) $\| R_n \|^{1/n} \to 0$ as $n \to \infty$.

Proof : Suppose (i) and (ii) holds good.

Let $x = < x_k > \in \overline{l_\infty\,(X, p)}$

$\Rightarrow \exists$ a real number $M > 0$ such that $\| x_k \|^{1/p_k} < M$.

since, $\| R_n \|^{1/n} \to 0$, and for every $\in > 0$,

$\| R_k \|^{1/k} < \in / M$ for sufficiently large k.

we have the identity by the Lemma (1),

$\left\| \sum_{k=n}^{n+p} A_k (x_k) \right\| \leq \| R_n \|$ .max $\{\| x_k \| : n \leq k \leq n + p\} \leq \in^k$

for sufficiently large k.

Hence $\Sigma A_k x_k$ converges.

consequently, $(A_k) \in \overline{l_\infty\,(X, p)}^{(m,\beta)}$

conversely, Let $< A_k > \in \overline{l_\infty\,(X, p)}^{(m,\beta)}$

since $\overline{l_\infty\,(X, p)} \subset \overline{l_\infty\,(X, p)}$

$\Rightarrow \overline{l_\infty\,(X, p)}^{(m,\beta)} \subset \overline{l_\infty\,(X, p)}^{(m,\beta)}$

and therefore $(A_k) \in \overline{l_\infty\,(X, p)}^{(m,\beta)} \Rightarrow (A_k) \in \overline{l_\infty\,(X, p)}^{(m,\beta)}$ which proves (1)

Now suppose, if possible that (ii) fails

Let $\lim_n \sup \| R_n \|^{1/n} = 3p > 0$.

Then, there exist natural numbers $n_1 > m_1 > m$ and z_m, ..., z_n in S such that

$$\left\| \sum_{k=m_1}^{n_1} A_k z_k \right\| > p$$

choose $m_2 > n_1$ such that $\| R_{m2} \|^{1/m2} > 2p$.

Then there exist n_2 ▯ m_2 and z_{m2}, ... ,z_{n2} in S such that $\left\| \sum_{m_2}^{n_2} A_k (z_k) \right\| > p$. Proceeding in this way, we and define

$x_k = 0$, $k \leq m1$

$= z_k$, $m1 \leq k \leq n1$

$= 0$, $n1 \leq k \leq m2$

$= z_k$, $m2 \leq k \leq n2$ and so on.

Then $x = (x_k) \in \overline{l_\infty\,(X, p)}$

But $\Sigma A_k x_k$ diverges which is contradiction to the fact that $(A_k) \in \overline{l_\infty\,(X, p)}^{(m,\beta)}$

Hence our assumption is wrong.

Therefore $\| R_n \|^{1/n} \to 0$ as $n \to \infty$.

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