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Non-Markovian Queueing System, M^x/G/1 with Server Breakdown and Repair Times

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Abstract

This paper deals with the steady state behavior of an MX/G/1 queue with breakdown. It assumed that customers arrive to the system in batches of variable size, but serve one by one. The main new assumption in this paper is that the repair process does not start immediately after a breakdown and there is a delay time waiting for repairs to start. We obtain steady state results in explicit and closed form in terms of the probability generating functions for the number of customers in the queue, the average waiting time in the queue.

Key Words: Stationary queue size distribution, Random breakdown, Delay times, Repair times

Introduction

In recent years, queue with server breakdown have emerged as an important area of queueing theory and have been studied extensively and successfully due to their various applications in production, communication systems. Segmented message transmission is practical applications of retrial queue in real life .We consider an M/G/1 retrial queue where blocked jobs on finding the server busy or broken down leave the service area and enter the retrial group in accordance with FCFS discipline. Several studies on retrial queues have been made by many researchers working in the area of applied probability theory from time to time. We assume that only the job at the head of the queue is allowed to occupy the server if it is free. The general retrial time policy arises naturally in many congestion problems related to many service systems where, after each service completion, the server spends a random amount of time in order to find the next job to be processed.

In many waiting line system, the role of server is played by mechanical /electronic device, such as computer, pallets, ATM, traffic light, etc., which is subjected to accidental random failures; until the failed server is repaired. Gharbi and Ioualalen[1] gave a detailed analysis of finite source retrial systems with multiple severs subject to random breakdown and repairs using generalized stochastic model and showed how this model is capable to cope up with the complexity of such retrial system involving the unreliability of the servers. In most of the queueing literature, it is assumed that the server is available in the service station on a permanent basis and service station never fails. However, these assumptions are particularly unrealistic. In practical system, we often meet the case where service stations may fail and can be repaired Similarly, many phenomena always occur in the area of computer communication networks and flexible manufacturing system,etc.Because the performance of such a system may be heavily affected by service station breakdown and delay in repair due to non availability of the repairman or of the apparatus needed for the repairs, such systems with a repairable service station are well worth investigating from the queueing theory viewpoint as well as reliability point of view.

Recently, there have been several contributions considering system of M/G/1 type in which the server may provide a second phase service. More specifically, we can analyze a system where customer's service may be viewed as scheduled in two phases: that is all the customers are processed in the first phase and only the customers who qualify are routed in the second phase. Such queueing situations occur in day –to –day life, for example, in a manufacturing process, all the arriving customers require the main service and only some of them may require the subsidiary service provided by the server. Some examples of queueing situations where such service mechanism can arise are also given. Madan[2-3] studied an M/G/1 queue with second optional service in which first essential service time follows a general distribution but second optional service is assumed to be exponentially distributed.. Choudhury[4-5] investigated this model in depth and generalized for batch arrival.Choudhury and Paul[6] investigated such a model under Bernoulli feedback mechanism. In this context, Krishna Kumar and Arivudainambi

[7] obtained the explicit expression for transient probabilities for this type of finite capacity model M/G/1/1.

On the other hand, many authors have investigated the system with a repairable service station wherein the service channel is subject to breakdowns or some other kinds of service interruption, which are beyond control of the server or the management. In most of the papers including the ones mentioned above the authors assume that whenever the system breaks down, the repair process starts immediately. In this context, recently, Ke and Pearn[8] have discussed an optional management policy for a Markovian mode.However, this is not the case in many real life situations. It is usually a common phenomenon that as a result of a sudden breakdown , the system has to wait for repair to start .We term this waiting time as 'Delay time' and the focus of this paper is to study the effect of this 'delay time' on the system among some earlier papers on service interruption.

This vacation period has been assumed to have general distribution. On the termination of a vacation, the server immediately joins the system to server the waiting customers. The 'delay time' before the repair processes as well as the 'repair time' after the delay are both assumed to have a general distribution. It is further assumed that the system starts working immediately after its repairs are complete. In addition to optional server vacations, the system suffers random breakdown from time to time.

Madhu Jain[9] investigated single unreliable server queueing model by incorporating Bernoulli feedback, general retrial time, K-phase optional repair along with modified vacation policy.

Queueing system with repeated attempts (retrials) are characterized by the feature that a customer who finds the server busy upon arrival, is obliged to leave the service area and to repeat his demand after some random amount of time called retrial time. Between trials, the blocked customer joins a pool of unsatisfied customers called "orbit". Queues in which customers are allowed to conduct retrials, have been widely used to model many practical problems in telephone switching systems, telecommunication networks and computers competing to gain service from a central processing unit. Moreover, retrial queues are also used as mathematical models of several computer systems; packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks etc. For a review of main results and methods, the reader is referred to the survey papers by Yang and Templeton[10].

In the current work, we consider queueing system with random breakdowns. Most of the papers on queues deal with either server vacations or system breakdowns. Very rare papers consider breakdowns in the system and those papers (Ke,[11]) rely on different assumptions for the queueing systems than those considered in the current paper.

In this paper, we present an analysis of the steady state behavior of a queueing system where breakdowns may occur at random and once the system breaks down, we assume that the service times, repair times and delay times each have a general distribution while the time to breakdown is exponential distribution.

Mathematical Model

We consider a $M^X/G/1$ queueing system where customers arrive according to a compound poisson process with mean arrival rate λ. Let $\lambda a_i t \Delta t (i=1,2,3......)$ be the first order probability that a batch of I customers arrives at the system during a short interval of time $(t, t+\Delta t)$, where $0 \le a_i \le 1 \, and \sum_{i=1}^{\infty} a_i = 1 and \lambda > 0$ is the arrival rate of batches. The customers are provided service one by one on a ' first come – first served basis' There is a single server and the service time follows a general (arbitrary) distribution with distribution function G(s) and density function g(s) . Let $\mu(x)\Delta x$ be the conditional probability density of service completion during the interval, $(x, x+\Delta x]$, given that the elapsed service time is x, so that

And therefore $\mu(x) = \dfrac{g(x)}{1-G(x)}$ .... (1)

On completion of service , the server may take a vacation of random length with

$$g(s) = \mu(s) e^{-\int_0^s \mu(x)dx} \quad .... (2)$$

The server's vacation time follows a general distribution with distribution function B(v) and density function b(v). Let $\beta(x)\Delta x$ be the conditional probability of a completion of a vacation during the interval $(x, x+\Delta x]$ given that the elapsed vacation time is x, so that

$$\beta(x) = \frac{b(x)}{1-B(x)} \quad .... (3)$$

$$b(v) = \beta(v) e^{-\int_0^v \beta(x)dx} \quad .... (4)$$

The system may breakdown at random, and breakdowns are assumed to occur according to a poisson stream with mean breakdowns assumed to occur according to a poisson stream with mean breakdown rate $\alpha > 0$. Further, we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of queue. Once the system breaks down, its repairs do not start immediately and there is a delay time. The delay time follows a general distribution with distribution function W(x) and density function w(x). Let $\phi(x)\Delta x$ be the conditional probability of a completion of a delay during the interval $(x, x+\Delta x]$ given that the elapsed delay time is x , so that

$$\phi(x) = \frac{w(x)}{1-W(x)} \quad .... (5)$$

and therefore

$$w(x) = \phi(x) e^{-\int_0^x \phi(t)dt} \quad .... (6)$$

The duration of repair follows a general distribution with distribution function H(r) and density function h(r). Let $\gamma(x)\Delta x$

$$\gamma(x)=\frac{h(x)}{1-H(x)} \quad \text{.........} \ (7)$$

$$h(x)=\gamma(x)e^{-\int_0^r \gamma(x)dx} \quad \text{.........} \ (8)$$

All stochastic processes involved in the system are independent of each other.

We assume that service's life time is exponentially distributed with rate α .when the server fails, it is immediately send for repair at a repair facility the repair facility require some time before starting the repair.

Notations

1. $P_n(x,t)$ **:** Probability that at time t , the server is active providing service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x . $P_n(t)=\int_0^\infty P_n(t,x)dx$ Denotes the probability that at time t there are n customers in the queue excluding the one customer in service irrespective of the value of x.

2. $V_n(x,t)$ **:** Probability that at time t, the server is on vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers waiting in the queue for service. $V_n(t)=\int_0^\infty V_n(t,x)dx$

Denotes the probability that at time t there are n customers in the queue and the server is on vacation irrespective of the value of x.

3. $R_n(t)$ **:** Probability that at time t, the server is inactive due to system breakdown and the system is under repair, while there are n $(n \geq 0)$ customers in the queue.

4. Q (t): Probability that at time t, there are no customers in the system and the server is idle idle but available in the system.

$$P_n(x)=\lim_{t\to\infty} p(t,x)$$

$$P_n=\lim_{t\to\infty}\int_0^\infty P_n(t,x)dx=\lim_{t\to\infty}P_n(t)$$

$$\lim_{t\to\infty} V_n(t,x)=V_n(x) \quad , \quad V_n=\lim_{t\to\infty}\int_0^\infty V_n(t,x)dx=\lim_{t\to\infty}V_n(t)$$

$$R_n(x)=\lim_{t\to\infty} R_n(t,x) \ ,$$

$$R_n=\lim_{t\to\infty}\int_0^\infty R_n(t,x)dx=\lim_{t\to\infty}R_n(t)$$

$$D_n(x)=\lim_{t\to\infty} D_n(t,x) \ ,$$

$$D_n=\lim_{t\to\infty}\int_0^\infty D_n(t,x)dx=\lim_{t\to\infty}D_n(t)$$

$$Q_n=\lim_{t\to\infty} Q_n(t)$$

$$\lim_{t\to\infty}\frac{dP_n(t)}{dt}=0,\ \lim_{t\to\infty}\frac{dV_n(t)}{dt}=0,\ \lim_{t\to\infty}\frac{dR_n(t)}{dt}=0,\ \lim_{t\to\infty}\frac{dD_n(t)}{dt}=0,\ n\geq 0 \quad \text{.......} \ (9)$$

Steady state distribution

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior.

$$\frac{\partial}{\partial x}P_n(x)=-\left(\lambda+\mu(x)+\alpha\right)P_n(x)+\lambda\sum_{i=1}^{n-1}a_iP_{n-i}(x) \ldots \ (10)$$

$$\frac{\partial}{\partial x}P_0(x)=-\left(\lambda+\mu(x)+\alpha\right)P_0(x) \quad \ldots \ (11)$$

$$\frac{\partial}{\partial x}V_n(x)=-\left(\lambda+\mu(x)+\alpha\right)V_n(x)+\lambda\sum_{i=1}^{n-1}a_iV_{n-i}(x) \ n\geq 1 \quad \ldots \ (12)$$

$$\frac{\partial}{\partial x}V_0(x)=-\left(\lambda+\beta(x)\right)V_0(x) \quad \ldots \ (13)$$

$$\frac{\partial}{\partial x}D_n(x)=-\left(\lambda+\phi(x)\right)D_n(x)+\lambda\sum_{i=1}^{n-1}a_iD_{n-i}(x) \ n\geq 1 \quad \ldots \ (14)$$

$$\frac{\partial}{\partial x}D_0(x)=0 \quad \ldots \ (15)$$

$$\lambda Q=\int_0^\infty R_0(x)\gamma(x)dx+\int_0^\infty V_0(x)\beta(x)dx+\int_0^\infty P_0(x)\mu(x)dx \quad \ldots \ (16)$$

$$\frac{\partial}{\partial x}R_n(x)+(\lambda+\gamma(x))R_n(x)=\lambda\sum_{i=1}^{n}a_iR_{n-i}(x) \ \ n\geq 1 \quad \ldots \ (17)$$

$$\frac{\partial}{\partial x}R_0(x)=-\left(\lambda+\gamma(x)\right)R_0(x) \quad \ldots \ (18)$$

The following boundary conditions will be used to solve the above equations

$$P_n(0)=\int_0^\infty P_{n+1}(x)\mu(x)dx+\int_0^\infty V_{n+1}(x)\beta(x)dx+\int_0^\infty R_{n+1}(x)\gamma(x)dx+\lambda a_{n+1}Q \quad n\geq 0 \quad \ldots \ (19)$$

$$D_n(0)=\alpha\int_0^\infty P_{n-1}(x)dx \quad n\geq 1 \quad \ldots \ (20)$$

$$D_n(0)=\alpha P_{n-1}$$

$$D_0(0)=0 \quad \ldots \ (21)$$

$$R_n(0)=\int_0^\infty D_n(x)\varphi(x)dx, \quad n\geq 0 \quad \ldots \ (22)$$

Queue size distribution at a random epoch

We define the following Probability generating functions

$$P_q(x,z)=\sum_{n=0}^{\infty} z^n P_n(x), \qquad P_q(z)=\sum_{n=0}^{\infty} z^n P_n,$$

$$V_q(x,z)=\sum_{n=0}^{\infty} z^n V_n(x), \qquad V_q(z)=\sum_{n=0}^{\infty} z^n V_n,$$

$$D_q(x,z)=\sum_{n=0}^{\infty} z^n D_n(x), \qquad D_q(z)=\sum_{n=0}^{\infty} z^n D_n,$$

$$R_q(x,z)=\sum_{n=0}^{\infty} z^n R_n(x), \qquad R_q(z)=\sum_{n=0}^{\infty} z^n R_n,$$

$$X(z)=\sum_{i=1}^{\infty} z^i q_i \qquad \text{...(23)}$$

We multiply equation (10) by z^n , take summation over n from 1 to ∞ ,adding to (11) then by simplifying and using equation (22) we get

$$\frac{\partial}{\partial x}P_q(x,z)+(\lambda-\lambda X(z)+\mu(x)+\alpha)P_q(x,z)=0 \qquad \text{.... (24)}$$

Following similar process, from equation (12), (13), (14), (15), (17), and (18) we get respectively

$$\frac{\partial}{\partial x}V_q(x,z)+(\lambda-\lambda X(z)+\beta(x))V_q(x,z)=0 \qquad \text{...(25)}$$

$$\frac{\partial}{\partial x}D_q(x,z)+(\lambda-\lambda X(z)+\varphi(x))D_q(x,z)=0 \qquad \text{... (26)}$$

$$\frac{\partial}{\partial x}R_q(x,z)+(\lambda-\lambda X(z)+\gamma(x))R_q(x,z)=0 \qquad \text{... (27)}$$

Multiply equation (19) by z^{n+1} , sum over n from 0 to ∞ , and use the generating functions defined in (24), we get

$$zP_q(0,z)=\int_0^\infty P_q(x,z)\mu(x)dx+\int_0^\infty V_q(x,z)\beta(x)dx+\int_0^\infty R_q(x,z)\gamma(x)dx$$
$$+\lambda X(z)Q-[\int_0^\infty P_0(x,z)\mu(x)dx+\int_0^\infty V_0(x,z)\beta(x)dx+\int_0^\infty R_0(x,z)\gamma(x)dx] \qquad \text{.... (28)}$$

Using equation (16) to replace

$$-[\int_0^\infty P_0(x)\mu(x)dx+\int_0^\infty V_0(x)\beta(x)dx+\int_0^\infty R_0(x)\gamma(x)dx] by \ -Q\lambda$$

we have

$$zP_q(0,z)-[\int_0^\infty P_q(x)\mu(x)dx+\int_0^\infty V_q(x)\beta(x)dx+\int_0^\infty R_q(x)\gamma(x)dx] \ +Q\lambda(X(z)-1) \qquad \text{... (29)}$$

Now multiply equation (20) by z^n and sum over n from 0 to ∞ , we get

$$D_q(0,z)=\alpha z P_q(z) \qquad \text{... (30)}$$

Now multiply equation (22) by z^n and sum over n from 0 to ∞ , we get

$$R_q(0,z)=\int_0^\infty D_q(x,z)\phi(x)dx \qquad \text{.......... (31)}$$

Integrating equation (24) from 0 to x yields

$$P_q(x,z)=P_q(0,z)e^{-(\lambda-\lambda X(z)+\alpha)x-\int_0^x \mu(t)dt} \qquad \text{.......... (32)}$$

Where $P_q(0,z)$ is given by equation (29)

Integrating equation (32) by parts with respect to x yields

$$P_q(z)=P_q(0,z)\left[\frac{1-G^*\left(\lambda-\lambda X(z)+\alpha\right)}{\left(\lambda-\lambda X(z)+\alpha\right)}\right] \qquad \text{.......... (33)}$$

$$G^*\left(\lambda-\lambda X(z)+\alpha\right)=\int_0^\infty e^{-(\lambda-\lambda X(z)+\alpha)x}dG(x)$$

is the Laplace –Stieltjes transform of the service time G(x).

Now multiplying both sides of equation (32) by $\mu(x)$ and integrating over x we get

$$\int_0^\infty P_q(x,z)\mu(x)dx=P_q(0,z)\,G^*\left(\lambda-\lambda X(z)+\alpha\right) \qquad \text{.......... (34)}$$

where $B^*\left[\lambda-\lambda X(z)\right]=\int_0^\infty e^{-(\lambda-\lambda X(z))x}dB(x)$

is the Laplace –Stieltjes transform of the vacation time B(x).

Integrating equation (26) from 0 to x, yields

$$D_q(x,z)=D_q(0,z)e^{-(\lambda-\lambda X(z))x-\int_0^x \phi(t)dt} \qquad \text{.......... (35)}$$

Substituting by the value of $D_q(0,z)$ from (30) in equation (35) we get

$$D_q(x,z)=\alpha z P_q e^{-(\lambda-\lambda X(z))x-\int_0^x \phi(t)dt} \qquad \text{.......... (36)}$$

Integrating equation (36) by parts with respect to x yields

$$D_q(z)=\frac{\alpha z P_q(z)\left(1-W^*(\lambda-\lambda X(z))\right)}{(\lambda-\lambda X(z))} \qquad \text{.......... (37)}$$

$W^*(\lambda-\lambda X(z))=\int_0^\infty e^{-(\lambda-\lambda X(z))x}dW(x)$ is the Laplace –Stieltjes transform of the Delay time B(x).

$$D_q(z)=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda X(z)+\alpha)\right)\left(1-W^*(\lambda-\lambda X(z))\right)}{(\lambda-\lambda X(z))(\lambda-\lambda X(z)+\alpha)} \qquad \text{.......... (38)}$$

Now multiplying both sides of equation (36) by $\phi(x)$ and integrating over x we

$$\int_0^\infty D_q(x,z)\phi(x)dx=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda X(z)+\alpha)\right)\left(1-W^*(\lambda-\lambda X(z))\right)}{(\lambda-\lambda X(z)+\alpha)} \qquad \text{...... (39)}$$

Now integrating equation (27) from 0 to x , yields

$$R_q(x,z)=R_q(0,z)e^{-(\lambda-\lambda X(z))x-\int_0^x \gamma(t)dt}$$

.... (40)

From equation (31) & (39) and putting the value $R_q(0,z)$ in equation (40) we obtain

$$R_q(x,z)=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda X(z)+\alpha)\right)\left(W^*(\lambda-\lambda X(z))\right)}{(\lambda-\lambda X(z)+\alpha)}e^{-(\lambda-\lambda X(z))x-\int_0^x \gamma(t)dt}$$

.... (41)

Now taking by parts integrating equation (41) with respect to x we obtain

$$R_q(z)=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda X(z)+\alpha)\right)\left(W^*(\lambda-\lambda X(z))\right)\left(1-H^*(\lambda-\lambda X(z))\right)}{(\lambda-\lambda X(z)+\alpha)(\lambda-\lambda X(z))}$$

.... (42)

Multiplying by $\gamma(x)$ both sides of equation (41) and integration with respect to x we obtain

$$\int_0^\infty R_q(x,z)\gamma(x)dx=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda X(z)+\alpha)\right)\left(W^*(\lambda-\lambda X(z))\right)\left(H^*(\lambda-\lambda X(z))\right)}{(\lambda-\lambda X(z)+\alpha)}$$

.... (43)

Assuming a = $\left(\lambda-\lambda X(z)+\alpha\right)$, $b=\left(\lambda-\lambda X(z)\right)$

Using equation (34) and (43) in equation (29) we obtain

$$P_q(z,0)=\frac{-abQ}{a\left(z-G^*(a)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)}$$

.... (44)

Substituting $P_q(z,0)$ in equation (32) ,(38) and (42) we get

$$P_q(z)=\frac{-bQ\left(1-G^*(a)\right)}{a\left(z-G^*(a)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)}$$

.... (45)

$$D_q(z)=\frac{-\alpha zQ\left(1-G^*(a)\right)\left(1-W^*(b)\right)}{a\left(z-G^*(a)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)}$$

.... (46)

$$R_q(z)=\frac{-\alpha zQ\left(1-G^*(a)\right)\left(1-H^*(b)\right)W^*(b)}{a\left(z-G^*(a)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)}$$

.... (47)

Let $S_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system

$$S_q(z)=P_q(z)+D_q(z)+R_q(z)$$

Then adding equations (45), (46) and (47) we obtain

$$S(z)=\frac{-Q\left(1-G^*(a)\right)\left\{b+\alpha z\left(1-W^*(b)H^*(b)\right)\right\}}{a\left(z-G^*(a)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)}$$

.... (48)

In order to find Q, we use the normalization condition

$$S_q(1)+Q=1$$

Note that for z =1, $S_q(1)$ is 0/0 form. Therefore, we apply L'Hopitals Rule on equation (48) we get

$$S(1)=\frac{\lambda E(I)Q\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)+\alpha E(D)\right)}{\alpha-\lambda E(I)\left(\left(1-G^*(\alpha)\right)\right)\left(1+\alpha E(R)+\alpha E(D)\right)-\alpha\left(1-G^*(\alpha)\right)}$$

.... (49)

Therefore, adding Q to equation (49) and equating to 1 simplifying we get

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)-E(D)\right)$$

.... (50)

Equation (50) gives the probability that the server is idle .From equation (50)the utilization factor , ρ of the system is given by

$$\rho=\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)-E(D)\right)$$

.... (51)

Where ρ <1 is the stability condition under which the steady states exits. Substituting for Q from (50) into (48), we have completely and explicitly determined $S_q(z)$, the probability generating function of the queue size.

Some Cases

Case 1: Exponential repair time.

For this distribution, the rate of service, rate of service $\gamma>0$, and we have

$$H^*(b)=\frac{\gamma}{b+\gamma} \quad , \quad E(R)=\frac{1}{\gamma}$$

.... (52)

Now using equation (48), (50) and (51) respectively we obtain

$$S_q(z)=\frac{-Q\left(1-G^*(a)\right)\left(cb+\alpha z\left(c-\gamma W^*(b)\right)\right)}{az\left(z-G^*(a)\right)-\alpha\gamma z\left(1-G^*(a)\right)W^*(b)}$$

.... (53)

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{1}{\gamma G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-\frac{1}{\gamma}-E(D)\right)$$

.... (54)

$$\rho=\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{1}{\gamma G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-\frac{1}{\gamma}-E(D)\right)$$

.... (55)

The results obtain in (53), (54) and (55) agree with the result given by the same authors in previous study.

Case 2:. No Delay for Repairs to start

Once the system breakdown, if its repairs start immediately and there is no delay time we let E (D) = 0 and W*(b) =1Using this in the main results of this paper,

$$S(z)=\frac{-Q\left(1-G^{*}(a)\right)\left\{b+\alpha z\left(1-H^{*}(b)\right)\right\}}{a\left(z-G^{*}(a)\right)-\alpha z\left(1-G^{*}(a)\right)H^{*}(b)}$$

………. (56)

$$Q=1-\lambda E\left(I\right)\left(\frac{1}{\alpha G^{*}(\alpha)}+\frac{E(R)}{G^{*}(\alpha)}-\frac{1}{\alpha}-E(R)\right)$$

………. (57)

$$\rho=\lambda E\left(I\right)\left(\frac{1}{\alpha G^{*}(\alpha)}+\frac{E(R)}{G^{*}(\alpha)}-\frac{1}{\alpha}-E(R)\right)$$

………. (58)

The results obtain in (53), (54) and (55) agree with the result given by Madan et al [12].

The Average Queue Size and the Average Waiting Time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q=\frac{d}{dz}S_q(z)\big|_{z=1}$$

……. (59)

Since this formula gives 0/0 form, then we write $S_q(z)$ given in (48) as $S_q(z)=\frac{N(z)}{D(z)}$

Where N(z) and D(z) are the numerator and denominator of the right hand side of (48) respectively .Then using the L' Hospital's rule twice we obtain

$$L_q=\lim_{z\to 1}\frac{D'(z)N''(z)-N'(z)D''(z)}{2\left(D'(z)\right)^2}$$

……. (60)

$$N'(1)=Q\lambda E\left(I\right)\left(1-G^{*}(a)\right)\left(1+\alpha\left(E(D)+E(R)\right)\right)$$

……. (61)

$$\begin{aligned}&N''(1)\\&=Q\alpha\left(\lambda E(I)\right)^2\left(1-G^{*}(a)\right)\left[E(D^2)+E(R^2)+E(D)+E(R)+2E(D)E(R)\right]\\&+Q\left(1-G^{*}(a)\right)\left[\lambda E(I)\left(E(D)-E(R)\right)(1+\alpha)\right]+QE\left(I(I-1)\right)\left(1-G^{*}(a)\right)\\&+2QG^{*}(\alpha)\left(\lambda E(I)\right)^2\left[1-\alpha\left(\left(E(D)+E(R)\right)\right)\right]\end{aligned}$$

……. (62)

$$D'(1)=-\lambda E\left(I\right)\left(1-G^{*}(\alpha)\right)\left(1+\alpha E(R)+\alpha E(D)\right)+\alpha G^{*}(\alpha)$$

……. (63)

$$\begin{aligned}&D''(1)=-2\lambda E(I)\left(1-G^{*}(a)\left(\lambda E(I)+\alpha\right)\right)-\\&\alpha\left(1-G^{*}(a)\right)\left\{\left[E(D)+E(R)\right]\left(\lambda E\left(I(I-1)\right)+\lambda E(I)\right)+\left(\lambda E(I)\right)^2\left[2E(D)E(R)+E(D^2)+E(R^2)\right]\right\}\\&-\alpha\lambda E(I)\left(G^{*}(\alpha)\lambda E(I)+1-G^{*}(\alpha)\right)\left(E(D)+E(R)\right)-\lambda E\left(I(I-1)\right)\left(1-G^{*}(a)\right)\end{aligned}$$

E(I(I-1)) is the second factorial moment of the batch size of the batch size of arriving customers, and Q has been found (50) Where $E\left(V^2\right)$ is the second moment of the vacation time, $E\left(R^2\right)$ is the second moment of the repair time $E\left(D^2\right)$ is the second moment of the batch size of arriving customers.

Then if putting the values of $N'(1)$, $N''(1)$ $D'(1)$ and $D''(1)$ from we get L_q in closed form. Further, the mean waiting time of a customer could be found using $W_q=\frac{L_q}{\lambda}$

Numerical Illustration

For the purpose of a numerical illustration, we consider the service times, vacation times, delay times and repair times are exponentially distributed. We choose the following arbitrary values μ = 7, λ = 2, φ = 8 E (I) =1 γ = 3 E (I (I-1)) = 0 while α varies from

1 to 5. Table1 shows that as long as we increase value of α the server idle time, the mean queue size and mean waiting time of customers all decreases while the utilization factor increase.

Table1: Value of various queue characteristics

α	Q	S(1)	L(1)	ρ
1	.5833	.2797	.5594	.4167
2	.4524	.2428	.4856	.5476
3	.3214	.1851	.3702	.6786
4	.1905	.1156	.2312	.8095
5	.0595	.0376	.0752	.9405

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