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On a mathematical model of the dysfunctions in smooth pursuit eye tracking

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Abstract

A simple mathematical deterministic model of dysfunctions of eye- tracking is presented in this paper. The model is formulated as a second order nonlinear ordinary differential equation, incorporating non Hookesien cubic restoring force. Perturbation technique with the nonlinear restoring force coefficient as the perturbation parameter is employed for solving the basic nonlinear equation. Numerical estimation of the angular displacement and angular velocity is computed for a wide spectrum of the eye dysfunction. The significance of the effects of the time, frequency and amplitude of the exiting force on angular displacement and angular velocity has been discussed by adopting ANOVA technique. The table shows the critical levels of time (τ), amplitude (Γ) and frequency (Ω) have also been noted corresponding to 0.05 level of significance.

Keywords: Dysfunctions in smooth pursuit eye tracking, nonlinear differential equation, ANOVA

INTRODUCTION

Dysfunctions in smooth pursuit eye movement are frequently encountered in schizophrenia patients [3], and also in some individuals with disorders of their central nervous system may be due to generic reasons. [1-4], An observer viewing a periodically moving object with an angular velocity less than 30 degrees per second tracks it to stabilize the moving image on the retina. Eye movement is being recorded invariably by varieties of techniques that range from infrared reflectometry to electrooculographic recordings [1-2]. Normal healthy individuals would be able to track smoothly the target up to speeds of 30 degrees per second .In contrast to this, schizophrenic patients show impairments characterized by a small extraneous eye movements superimposed on a smooth tracking signal induced by the target periodicity [8].

Holzman [3] attributes such anomalies to central nervous system dysfunctions above the brain stem that also manifests itself in a disorder of involuntary attention. It is also felt that such disorders maybe of genetic origin .However no specific mechanism has been identified as responsible for such disorders .The absence of a satisfactory explanation for the observed eye dysfunction in schizophrenic patients warrants a proper analysis of the generic features of the problem of eye tracking .

This paper presents a humble start of a much wider investigation with a simple deterministic model of eye-dysfunctions in the form of a second order nonlinear Duffing type of ordinary differential equation incorporating a weak non-Hooksien cubic restoring force and a linear Newtonian damper. The initial angular displacement is taken to be zero together with no take off angular velocity. An approximate solution of the non homogeneous model equation is obtained by regular perturbation technique. The dysfunction dynamic parameters: angular displacement and angular velocity have been computed numerically for a wide spectrum of eye-dysfunction characteristics of schizophrenic patients [6]. ANOVA (Analysis of variance) has been carried out to highlight the significance of the effects of the amplitude and frequency of excitation at each instant of time on the angular displacement and angular velocity. Critical levels of these parameters at 0.05level of significance have been identified using linear interpolation. Variations of the angular displacement versus time and Variations of the angular velocity versus time for different values of the amplitude (Γ) and angular frequency (Ω) are illustrated (fig:2-10) and phase portraits (Displacement-Velocity orbits) for the different values of Amplitude (Γ), frequency (Ω) are illustrated (fig:11-16) MATLAB has been employed in tracing these illustrations.

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The model

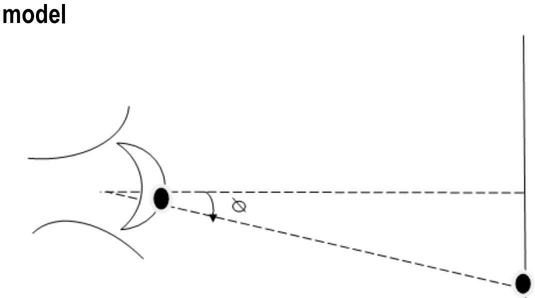

Fig 1. Schematic diagram of an eye tracking experiment

Consider a tracking object (eye) which can perform constrained rotations so as to follow a target moving periodically with a given angular velocity. (Fig I)

NOTATION ADOPTED

Ø :The angle between normal to the screen and the line connecting the target's position at time t. and the eye

I : Moment of inertia of the eye about the normal to the screen

α: Damping coefficient.
K : The Hooksien restoring constant.
L : A (nonlinear) non-hooksien restoring force coefficient (assumed to be small).
A : peak to peak amplitude of the target moving periodically.
ω_d : The frequency of the target (relative to the eye).

MATHEMATICAL MODEL EQUATION

Assumptions

1. Eye damping factor is proportional to its angular velocity (Newton's law).
2. The nonlinear restoring force of the eye is weak and assumed to be Non-
Hooksien-Duffing type.

The deterministic eye dynamics in the presence of target moving periodically can be characterized by the below equation

$$I\frac{d^2\phi}{dt^2}+\alpha\frac{d\phi}{dt}+k\phi-L\phi^3=A\cos(\omega_d t) \quad (1)$$

With the initial conditions

$$\phi(0)=0, and\ \phi^*(0)=0 \quad (2)$$

Employing the following non dimensional parameters

$$\frac{\phi}{\phi_0}=\psi,\ \frac{k}{I}=\omega_0^2,\ \frac{\omega_d}{\omega_0}=\Omega,\ \frac{\alpha}{I\omega_0}=2\delta,\ \frac{\phi_0^2 L}{I\omega_0^2}=\varepsilon,\ \frac{A}{I\omega_0^2\phi_0}=\Gamma \quad (3)$$

ε is the parameter characteristics of the non hooksien restoring force

The equation (1) can be written as

$$\frac{d^2\psi}{d\tau^2}+2\delta\frac{d\psi}{d\tau}+\psi-\varepsilon\psi^3=\Gamma\cos(\Omega\tau) \quad (4)$$

with initial conditions: no-initial angular displacement and no-initial angular velocity, i.e.

$$\psi(0)=0,\psi^*(0)=0 \quad (5)$$

It can be noted that δ is the damping parameter. For simplicity of analysis, the case δ=1 is investigated in this present paper. The cases in which for δ=0 (undamped case) and δ ≠1 will be considered separately in the forth coming communications. An approximate solution of the nonlinear equation (4) with the conditions (5) is sought for small values of ε ≤ 1 taken as the perturbation parameter. Following Nyfeh&Mook [7] $\psi(\tau)$ is expressed as

$$\psi(\tau)=\psi^{(0)}(\tau)+\varepsilon\psi^{(1)}(\tau)+\varepsilon^2\psi^{(2)}(\tau)+--- \quad (6)$$

Substituting (6) in the equation (4) and collecting like powers of ε, we obtain the basic equations in various orders of approximation.

Basic solution (**Collecting the coefficients of ε°)** w get

$$\frac{d^2\psi^{(0)}}{d\tau^2}+2\frac{d\psi^{(0)}}{d\tau}+\psi^{(0)}=\Gamma\cos(\Omega\tau) \quad (7)$$

with the initial conditions

$$\psi^{(0)}(0)=0,\ \psi^{(0)*}(0)=0 \quad (8)$$

These equations yield the solution

$$\psi^{(0)}(\tau)=(c_1+c_2\tau)e^{-\tau}+\Gamma\cos^2\theta\cos(\Omega\tau-2\theta) \quad (9)$$

where $c_1=-\Gamma\cos^2\theta\cos(2\theta)$ and $c_2=-\Gamma\cos^2\theta$ where $\theta=\tan^{-1}(\Omega)$

First approximation: (Collecting the coefficient of ε^1)

$$\frac{d^2\psi^{(1)}}{d\tau^2}+2\frac{d\psi^{(1)}}{d\tau}+\psi^{(1)}=(\psi^{(0)})^3 \quad (10)$$

together with the initial conditions

$$\psi^{(1)}(0)=0,\ \psi^{(1)*}(0)=0 \quad (11)$$

The solution of the equation (10) satisfying the conditions (11) is

$$\begin{aligned}\psi^{(1)}(\tau) &= (c_3+c_4\tau)+\frac{e^{-3\tau}}{4}\left\{(c_1+c_2\tau)^3+3c_2(c_1+c_2\tau)^2+\frac{9}{2}c_2^2(c_1+c_2\tau)+3c_2^3\right\}\\ &+3\Gamma\cos^4\theta e^{-2\tau}\left[\begin{array}{l}\cos(\Omega\tau)\left[(c_1+c_2\tau)^2+4c_2\cos^2\theta\ (c_1+c_2\tau)+6c_2^2\cos^2\theta\cos2\theta\right]\\ -\ \sin(\Omega\tau)\left[2\sin2\theta c_2(c_1+c_2\tau)+6c_2^2\cos^2\theta\sin2\theta\right]\end{array}\right]\\ &+\frac{3\Gamma^2\cos^4\theta}{2}e^{-\tau}\left\{\frac{(c_1+c_2\tau)^3}{6c_2^2}\right\}\\ &+\frac{3\Gamma^2\cos^4\theta}{2}e^{-\tau}\left\{\frac{c_2\sin(\Omega\tau-4\theta)}{4\Omega^3}-\frac{(c_1+c_2\tau)\cos(\Omega\tau-4\theta)}{4\Omega^2}\right\}\\ &+\frac{3\Gamma^3\cos^8\theta}{4}\cos(\Omega\tau-4\theta)+\frac{1}{4}\Gamma^3\cos^6\theta\cos^2\alpha\cos(3\Omega\tau-6\theta-2\alpha)\end{aligned}$$

$$\alpha=\tan^{-1}(3\Omega) \quad (12)$$

Where

(13)

The approximate solution for the equation (3) up to O(ε) is given by

$$\psi(\tau)=\psi^{(0)}(\tau)+\varepsilon\psi^{(1)}(\tau) \quad (14)$$

$$\psi(\tau) = (c_1 + c_2\tau)e^{-\tau} + \Gamma\cos^2\theta\cos(\Omega\tau - 2\theta) +$$
$$\varepsilon\left\{\begin{array}{l}(c_3+c_4\tau)+\dfrac{e^{-3\tau}}{4}\left\{(c_1+c_2\tau)^3+3c_2(c_1+c_2\tau)^2+\dfrac{9}{2}c_2^2(c_1+c_2\tau)+3c_2^3\right\}\\ +3\Gamma\cos^4\theta e^{-2\tau}\left[\begin{array}{l}\cos(\Omega\tau)\left[(c_1+c_2\tau)^2+4c_2\cos^2\theta\,(c_1+c_2\tau)+6c_2^2\cos^2\theta\cos2\theta\right]\\ -\sin(\Omega\tau)\left[2\sin2\theta c_2(c_1+c_2\tau)+6c_2^2\cos^2\theta\sin2\theta\right]\end{array}\right]\\ +\dfrac{3\Gamma^2\cos^4\theta}{2}e^{-\tau}\left\{\dfrac{(c_1+c_2\tau)^3}{6c_2^2}\right\}\\ +\dfrac{3\Gamma^2\cos^4\theta}{2}e^{-\tau}\left\{\dfrac{c_2\sin(\Omega\tau-4\theta)}{4\Omega^3}-\dfrac{(c_1+c_2\tau)\cos(\Omega\tau-4\theta)}{4\Omega^2}\right\}\\ +\dfrac{3\Gamma^3\cos^8\theta}{4}\cos(\Omega\tau-4\theta)+\dfrac{1}{4}\Gamma^3\cos^6\theta\cos^2\alpha\cos(3\Omega\tau-6\theta-2\alpha)\end{array}\right\} \tag{15}$$

$$\psi^{\bullet}(\tau) = -(c_1+c_2\tau)e^{-\tau}+c_2e^{-\tau}-\frac{\Gamma}{2}\sin2\theta\sin(\Omega\tau-2\theta)+$$
$$\varepsilon\left\{\begin{array}{l}-(c_3+c_4\tau)e^{-\tau}+c_4e^{-\tau}\\ -e^{-3\tau}\left(\dfrac{3}{2}c_2(c_1+c_2\tau)^2+\dfrac{15}{8}c_2^2(c_1+c_2\tau)+\dfrac{9}{8}c_2^3+\dfrac{3}{4}(c_1+c_2\tau)^3\right)\\ +\Gamma\cos^4\theta e^{-2\tau}\left(\begin{array}{l}-3(c_1+c_2\tau)^2\{\tan\theta\sin\Omega\tau+2\cos\Omega\tau\}\\ +6c_2(c_1+c_2\tau)\{\cos(\Omega\tau+2\theta)-4\cos\theta\cos(\Omega\tau+\theta)\}\\ +c_2^2\{6\cos(\Omega\tau+2\theta)+\dfrac{9}{2}\cos(\Omega\tau+4\theta)\\ +\dfrac{3}{2}\cos\Omega\tau-36\cos^2\theta\cos(\Omega\tau+2\theta)\}\end{array}\right)\\ +\dfrac{3\Gamma^2\cos^4\theta e^{-\tau}}{4c_2^2}\left(3c_2(c_1+c_2\tau)^2-(c_1+c_2\tau)^3\right)\\ +\dfrac{3\Gamma^2\cos^3\theta e^{-\tau}}{8\sin^2\theta}\left(c_2\cot\theta\sin(5\theta-2\Omega\tau)+(c_1+c_2\tau)\cos(2\Omega\tau-5\theta)\right)\\ -\dfrac{3}{4}\Gamma^3\sin\theta\cos^7\theta\sin(\Omega\tau-4\theta)-\dfrac{3}{4}\Gamma^3\sin\theta\cos^5\theta\cos^2\alpha\sin(3\Omega\tau-6\theta-2\alpha)\end{array}\right\} \tag{16}$$

Where

$$c_1 = -\Gamma\cos^2\theta\cos(2\theta)\quad,\quad c_2 = -\Gamma\cos^2\theta$$
$$c_3 = -\Gamma^3\{68\cos^{12}\theta-45\cos^{10}\theta-\frac{3}{4}\cos^8\theta+\cos^6\theta(\frac{1}{8}+\frac{15}{8}\cot^2\theta+\frac{1}{4}\cos^2\alpha\cos(6\theta+2\alpha))\}$$
$$c_4 = -\Gamma^3\left[\begin{array}{l}4\cos^{12}\theta-36\cos^{10}\theta+\dfrac{69}{4}\cos^8\theta-\dfrac{199}{8}\cos^6\theta-\dfrac{99}{8}\cot^2\theta\cos^6\theta+\dfrac{1}{4}\cos^2\alpha\cos^6\theta\cos(6\theta+2\alpha)\\ -18\cos^{10}\theta\cot^2\theta+27\cot^2\theta\cos^8\theta+24\cos^4\theta\cot^2\theta+\dfrac{3}{4}\sin\theta\cos^5\theta\cos^2\alpha\sin(6\theta+2\alpha)\end{array}\right] \tag{17}$$

Numerical estimation of the angular displacement and angular velocity have been carried out , for a wide spectrum of values of Ω =0.1-(0.1)-0.5 and Γ =0.1-(0.1)-0.5 within the time interval 0 < $\tau \le$ 100. Critical values of these dysfunction parameters at which their effects on the variations in angular displacement and angular velocity would be significant at 5% level of tolerance have been identified by employing ANOVA technique and linear interpolation. The computational details are not included in the present communication due to space limitations. However, these values are stated in the conclusions. Variations of the angular displacement versus time and variations of angular velocity versus time for different values of the amplitude (Γ) and angular frequency (Ω) are illustrated (fig: 2-10) and phase portraits (Displacement-Velocity orbits) for the different values of Amplitude (Γ), frequency (Ω) are illustrated (fig: 11-16).

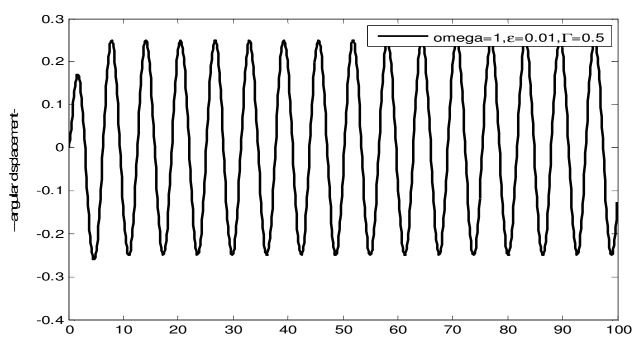

Fig II. variation of the angular displacement versus dimensionless time for the frequency Parameter (Ω=1) , amplitudes (Γ=0.5)and nonlinear restoring force coefficient(ε=0.01);

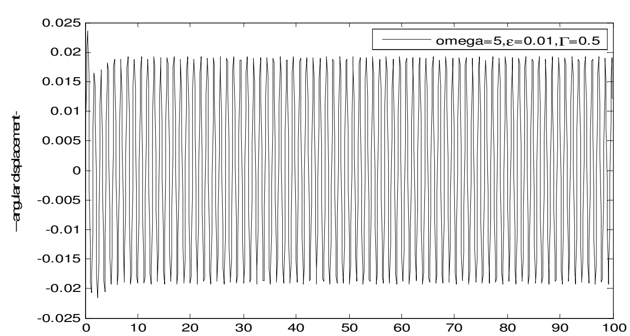

Fig III. variation of the angular displacement versus dimensionless time for the frequency Parameter (Ω=5), amplitudes (Γ=0.5) and nonlinear restoring force coefficient(ε=0.01);

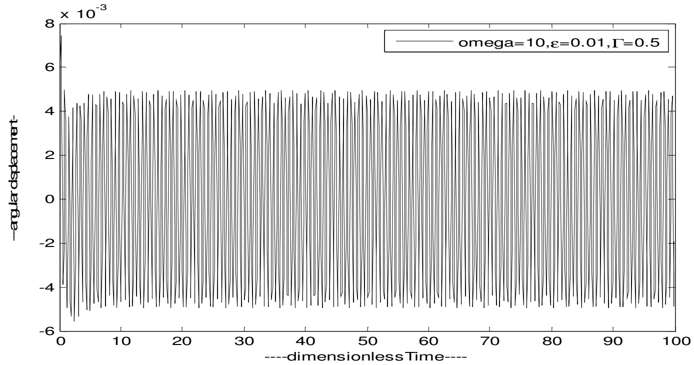

Fig iv. variation of the angular displacement versus dimensionless time for the frequency Parameter (Ω=10), amplitudes (Γ=0.5) and nonlinear restoring force coefficient(ε=0.01);

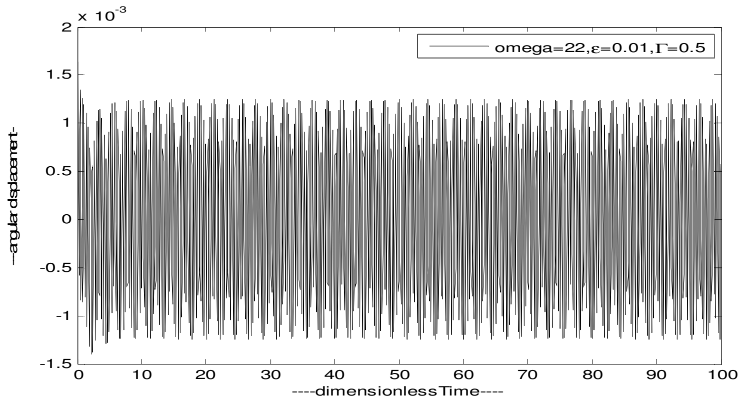

Fig v. variation of the angular displacement versus dimensionless time for the frequency Parameter (Ω=22), amplitudes (Γ=0.5) and nonlinear restoring force coefficient(ε=0.01);

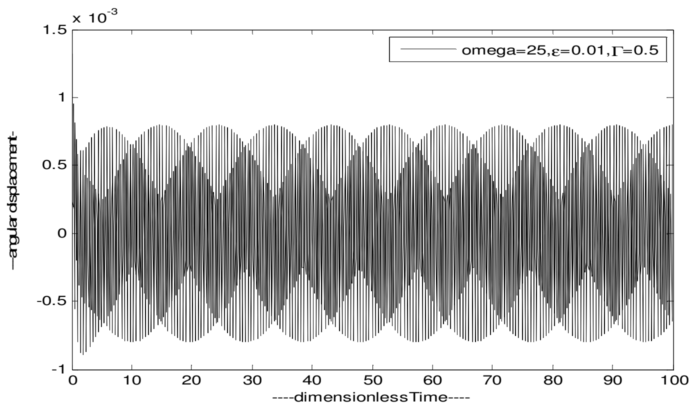

Fig vi. variation of the angular displacement versus dimensionless time for the frequency Parameter (Ω=22), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

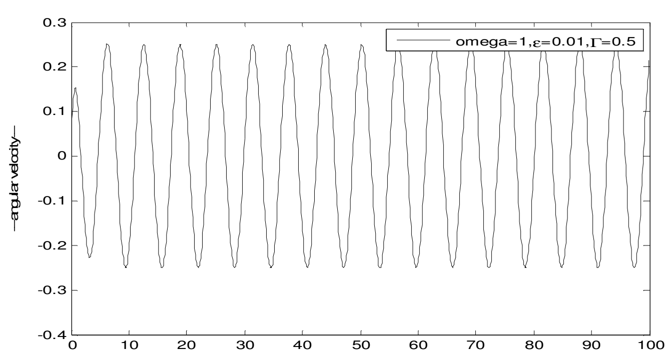

Fig vii. variation of the angular velocity versus dimensionless time for the frequency Parameter (Ω=1), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

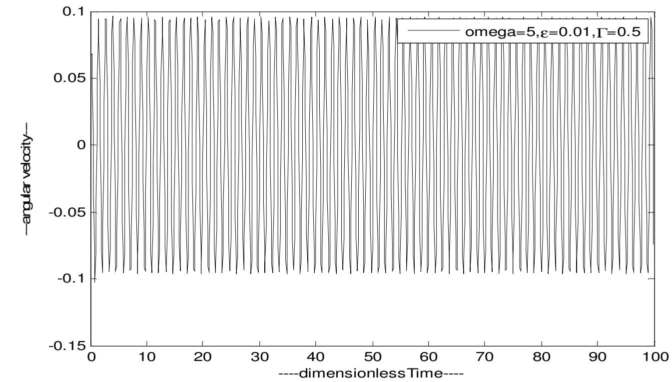

Fig viii. variation of the angular velocity versus dimensionless time for the frequency Parameter (Ω=5), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

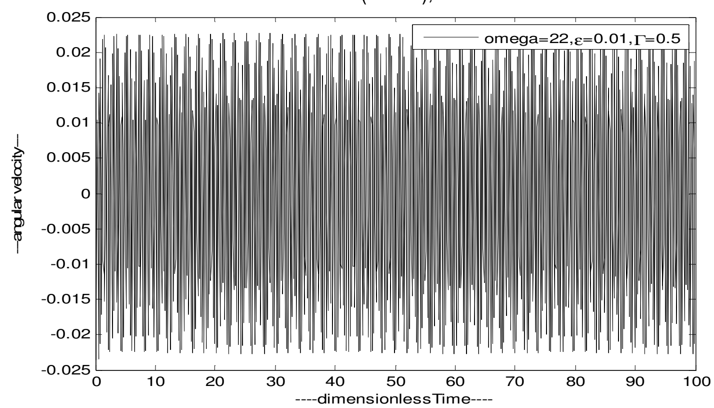

Fig ix. variation of the angular velocity versus dimensionless time for the frequency Parameter (Ω=22), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

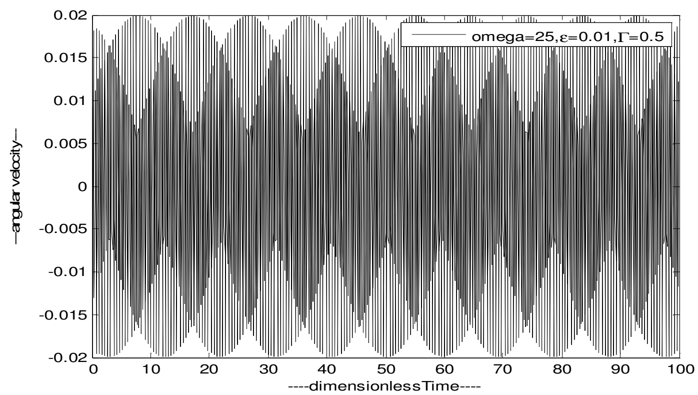

Fig x. variation of the angular velocity versus dimensionless time for the frequency Parameter (Ω=25), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

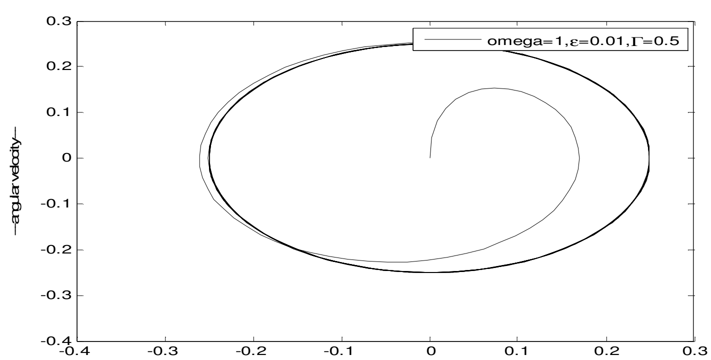

Fig xi. variation of the angular velocity versus angular displacement for the frequency Parameter (Ω=1), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

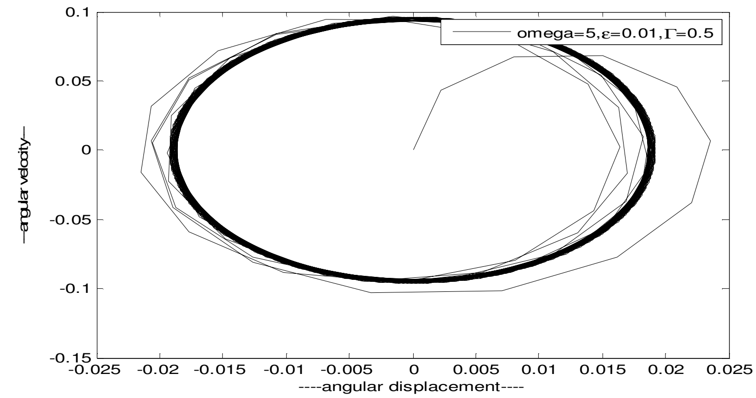

Fig xii. variation of the angular velocity versus angular displacement for the frequency Parameter (Ω=5), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

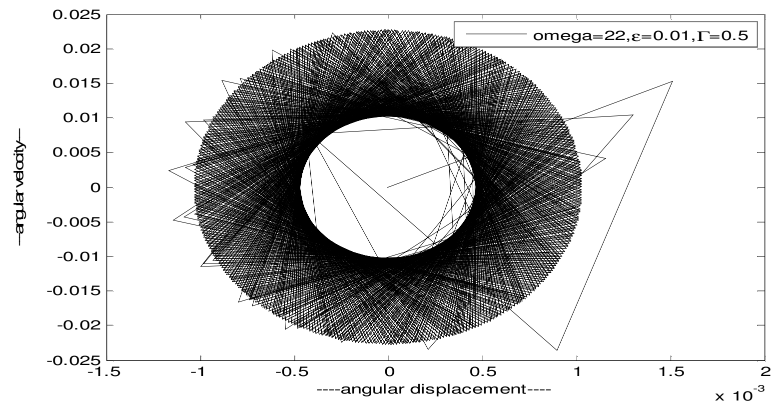

Fig xiii. variation of the angular velocity versus angular displacement for the frequency Parameter (Ω=22), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

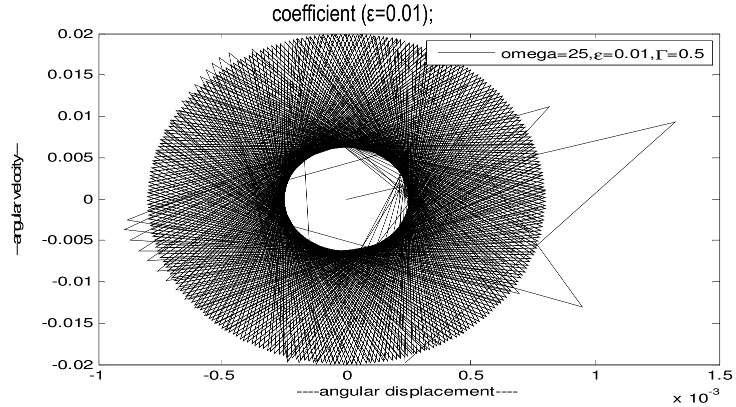

Fig xiv. variation of the angular velocity versus angular displacement for the frequency Parameter (Ω=25), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

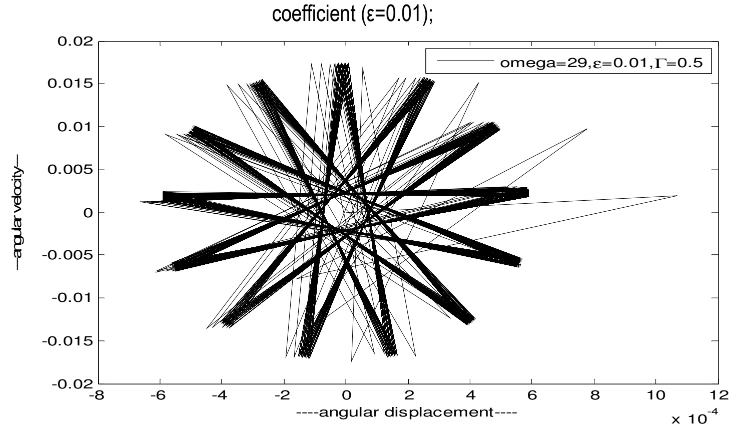

Fig xv. variation of the angular velocity versus angular displacement for the frequency Parameter (Ω=29), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01);

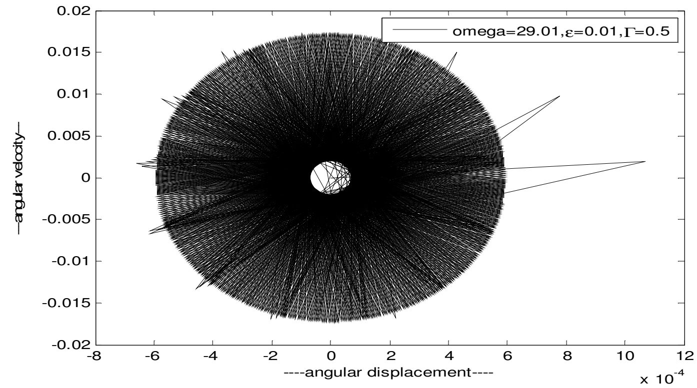

Fig xvi. variation of the angular velocity versus angular displacement for the frequency Parameter (Ω=29.01), amplitudes (Γ=0.5) and nonlinear restoring force coefficient (ε=0.01)

Based on the numerical computations carried out (the details of which are not shown here due to space limitations) the following conclusions are drawn.

CONCLUSIONS

(1) At different levels of time (τ) for fixed amplitude (Γ) and frequency Ω)

The variation in angular displacement of the eye due to amplitude- variation is significant up to the time instant **τ** =5.59sec, there after no appreciable change in angular displacement of the eye due to amplitude variations would be observed.

(2) At different levels of frequency (Ω) for fixed amplitude (Γ)and time(τ)

The variation in the angular displacement of the eye as time progresses is significant up to Ω =0.5795beyond which there would not be any significance in such variations.

(3)At different levels of amplitude (Γ) for fixed frequency (Ω) and time (τ)

The difference in angular displacement of the eye due to frequency- variations is significant up to Γ=5.125, beyond which no appreciable variations in angular displacement would be noticed.

(4) At different levels of time (τ) for fixed amplitude (Γ) and frequency Ω)

The variation in angular velocity of the eye due to amplitude- variation is significant up to the time instant **τ** =2.718sec, there after no appreciable change in angular velocity of the eye due to amplitude variations would be observed.

(5) At different levels of frequency (Ω) for fixed amplitude (Γ)and time(τ)

The variation in the angular velocity of the eye as time progresses is significant up to Ω =0.181149375beyond which there would not be any significance in such variations.

(6) At different levels of amplitude (Γ) for fixed frequency (Ω) and time (τ)

The illustrations exhibit the erratic variations of the angular displacement with the increase of Γ, Ω and τ of schizophrenia patient. Beyond the critical values, mentioned earlier, there would not be any variation leading to the arrest of the movements of the eye ball. The state of starring vision of a mentally retarded patient would begin at the time instant of the happening of the above events (1),(2)and(3) whichever is earlier. This would naturally depend upon movement of the target object and the constitution of the individual patient and the state of disease severity

The difference in angular velocity of the eye due to amplitude- variations is more significant compared to that due to the frequency variation. The illustrations exhibit the erratic variations of the angular velocity with the increase of Γ, Ω and τ of schizophrenia patient. Beyond the critical values, mentioned above, there would not be any variation leading to the arrest of the movements of the eye ball. The state of starring vision of a mentally retarded patient would begin at the time instant of the happening of the above events (1),(2)and(3) whichever is earlier. This would naturally depend upon movement of the target object and the constitution of the individual patient and the state of disease severity.

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