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Commensal –decay host ecosystem with a constant replenishment to the host species

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Abstract
The present paper deals with an analytical investigation on two species commensal-host ecological model with the host has Mortality rate and is being harvested (immigrated) at a constant rate. Further, both the species are with limited resources. This model is characterized by a couple of first order non-linear ordinary differential equation. All possible, two equilibrium points are identified and a stability criterion for it is discussed. Solutions for the linearized perturbed equations are found and results are presented.

Keywords: Equilibrium points, Equilibrium state, Stability, Carrying capacity, Reversal time of dominance

INTRODUCTION

An ecosystem is a complex set of relationships among living resources, habitats and residents of a region and ecology is the scientific study of the processes influencing the distribution and abundance of organisms, the interactions among organisms, and the interactions between organisms and the transformation and flux of energy and matter. The ecological interactions can be broadly classified as Prey-Predation, Competition, Commensalism, Ammensalism and Neutralism and so on. Research in theoretical ecology was initiated by Lotka [11] and by Volterra [17].Since then many Mathematicians and Ecologists contributed to the growth of this area of knowledge as reported in the treatises of Mayer [12], Paul Colinvaux [13], Kapur [6, 7], Svirezhev and Logofet [16], Freedman [5], Kushing [8]. N.C.Srinivas [15] studied the competitive ecosystems of two species and three species with limited and unlimited resources. Later Lakshminarayan and Pattabhi Ramacharyulu [9, 10] investigated prey-predator ecological models with a partial cover for the prey and alternative food for the predator and prey-predator model with cover for prey and alternate food for the predator and time delay. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Gandhi [1, 2], by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [3, 4], while the mutualism between two species was examined by Ravindra Reddy [14].

The present investigation is on a two species commensalism model with mortality rate for the host species and is being immigrated at a constant rate. The mathematical model is characterized by a couple of first order non-linear ordinary differential equations. The two existing equilibrium points are identified and a stability criterion for it is discussed. Solutions for the linearized perturbed equations are found and results are illustrated.

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BASIC EQUATIONS

Notation Adopted

$N_1(t)$	:The population of the commensal species *(S_1).*
$N_2(t)$	:The population of the host species *(S_2).*
a_1	:The rate of natural growth of the commensal*(S_1).*
d_2	:The rate of natural death of the host *(S_2).*
a_{11}	:The rate of decrease of commensal *(S_1).* due to the limitations of its natural resources.
a_{22}	:The rate of decrease of the host *(S_2)* due to the limitations of its natural resources.
a_{12}	:The rate of increase of the comminsal *(S_1).* due to the Support given by the host *(S_2).*
$k_1(=a_1/a_2)$	:The carrying capacity of S_1.
$c(= a_{12}/a_{11})$	:The Mortal coefficient of S_2.
$e_2 (=d_2/a_{22})$	:The Mortal coefficient of S_2.
H_2	:The renewal or replenishment of S_2 per unit time.

Further both the variables $N_1(t)$ and $N_2(t)$ are non-negative for all 't' and all the model parameters a_1, d_2, a_{11}, a_{12}, a_{22}, and H_2 are assumed to be non-negative constants.

Employing the above terminology, the model equations for Commensal-Mortal Host ecosystem where the host species is being harvested (immigration) at a constant rate is given by the following system of non-linear first order ordinary differential equations.

(i). Equation for the growth rate of the Commensal Species *S_1*:

$$\frac{dN_1}{dt} = a_{11}N_1\left[k_1 - N_1 + cN_2\right] \tag{2.1}$$

(ii). Equation for the growth rate of the Host Species *S_2*:

$$\frac{dN_2}{dt} = a_{22}\left[-e_2N_2 - N_2^{\,2} + H_2\right] \tag{2.2}$$

EQUILIBRIUM POINTS

The system under investigation has the following two-equilibrium point given by $\frac{dN_i}{dt} = 0,\ \ i = 1,2$.

E_1: $\overline{N_1} = 0$; $\overline{N_2} = \frac{-e_2 + \sqrt{e_2^{\,2} + 4H_2}}{2}$ (i.e. Commensal washed out state) (3.1)

E_2: $\overline{N_1} = k_1 + \frac{c\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]}{2}$;

$\overline{N_2} = \frac{-e_2 + \sqrt{e_2^{\,2} + 4H_2}}{2}$ (i.e. Co-existent State) (3.2)

THE STABILITY OF THE EQUILIBRIUM STATES

Let

$$N_1 = \left(N_1, N_2\right) = \overline{N_1} + \bar{u} \tag{4.1}$$

where $\bar{u} = (u_1, u_2)$. The perturbations u_1, u_2 over the equilibrium state $\bar{N} = (\overline{N_1}, \overline{N_2})$ are so small that their second and higher powers and products are negligible. The basic equations (2.1) and (2.2) are linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{4.2}$$

where $A = \begin{bmatrix} k_1 a_{11} - 2a_{11}\overline{N_1} + ca_{11}\overline{N_2} & ca_{11}\overline{N_1} \\ 0 & -e_2 a_{22} - 2a_{22}\overline{N_2} \end{bmatrix}$ (4.3)

The characteristic equation for the system is det

$$[A - \lambda I] = 0 \tag{4.4}$$

The equilibrium state is stable, only when both the eigenvalues are (i). negative in case they are real or (ii). have negative real parts in case they are complex.

Stability of the Equilibrium State E_1

In this case the corresponding perturbed equations are given by

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}\left(k_1 + \frac{c}{2}\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]\right) & 0 \\ 0 & -a_{22}\sqrt{e_2^{\,2} + 4H_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The corresponding characteristic equation is

$$\left(\lambda - a_{11}\left(k_1 + \frac{c}{2}\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]\right)\right)\left(\lambda + a_{22}\sqrt{e_2^{\,2} + 4H_2}\right) = 0 \tag{4.5}$$

the roots of which are

($\lambda_1 = a_{11}\left(k_1 + \frac{c}{2}\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]\right) > 0$ and $\lambda_2 = -a_{22}\sqrt{k_2^{\,2} + 4H_2} < 0$)

since one of the two roots is positive. Hence the study state is unstable.

By solving the system of perturbed equations, we get

$$u_1 = u_{10}\, e^{a_{11}\left(k_1 + \frac{c}{2}\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]\right)t};$$

$$u_2 = u_{20}\, e^{-\left(a_{22}\sqrt{e_2^{\,2} + 4H_2}\right)t} \tag{4.6; 4.7}$$

The solution curves in this case are illustrated and the conclusions are given below.

Case (i): When $u_{10} > u_{20}$

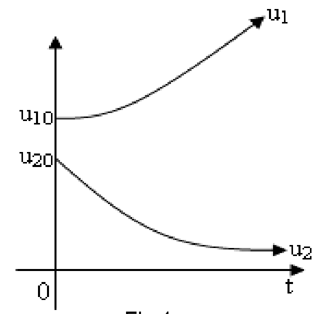


Fig 1.

Case (ii): When $u_{10} < u_{20}$

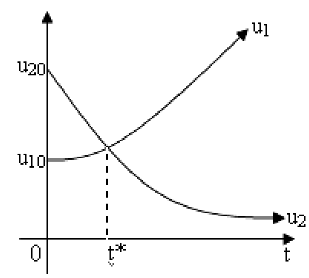


Fig 2.

Case (i)

The perturbation in the initial population strength of the commensal is greater than that of the host. In this case the commensal species is observed to move away from the equilibrium point as shown in Fig. 1, while the host species is moving towards the equilibrium point.

Case (ii)

In this case perturbations in the host out numbers the commensal till the time instant

$$t = t^* = \frac{1}{\left[a_{11}\left(k_1 + \frac{c}{2}\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]\right) + a_{22}\sqrt{k_2^{\,2} + 4H_2}\right]}\log\left(\frac{u_{20}}{u_{10}}\right)$$

this is the dominance reversal time as shown in Fig.2.

Trajectories of Perturbed Species

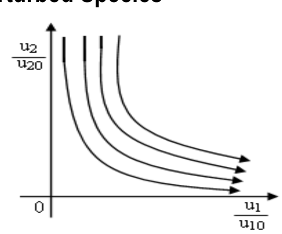


Fig 3.

Eliminating 't' between the equations (4.6) and (4.7), we obtain

$$\left(\frac{u_1}{u_{10}}\right)^{-a_{22}\sqrt{k_2^{\,2} + 4H_2}} = \left(\frac{u_2}{u_{20}}\right)^{a_{11}\left(k_1 + \frac{c}{2}\left[-e_2 + \sqrt{e_2^{\,2} + 4H_2}\right]\right)} \tag{4.8}$$

the resulting curves of (4.8) are hyperbolic and are shown in Fig.3.

Stability of the Equilibrium State *E₂*

The corresponding linearized perturbed equations in this state are

$$\frac{d}{dt}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}-a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right) & ca_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)\\ 0 & -a_{22}\sqrt{e_2^{\,2}+4H_2}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \quad (4.9)$$

The characteristic equation for the system (4.9) is

$$\left(\lambda+a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)\right)\left(\lambda+a_{22}\sqrt{e_2^{\,2}+4H_2}\right)=0 \quad (4.10)$$

Since both the roots $\lambda_1 = a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)$ and $\lambda_2=-a_{22}\sqrt{e_2^{\,2}+4H_2}$ of the equation (4.10) are negative, the steady *state is stable.*

The solutions of equations (4.8) are given by

$$u_1=\left[u_{10}-L_1\right]e^{-a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)t}+L_1e^{-\left(a_{22}\sqrt{e_2^{\,2}+4H_2}\right)t} \quad (4.11)$$

where

$$L_1=\frac{a_{11}cu_{20}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)}{a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)-a_{22}\sqrt{e_2^{\,2}+4H_2}} \quad (4.11.1)$$

$$u_2=u_{20}e^{-\left(a_{22}\sqrt{e_2^{\,2}+4H_2}\right)t} \quad (4.12)$$

It is to be noted that $a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\,2}+4H_2}\right]\right)>0$ and $a_{22}\sqrt{e_2^{\,2}+4H_2}>0$ and also noticed that $u_1\to 0$ and $u_2\to 0$ as $t\to\infty$.

There arise the following two cases:

Case A : $u_{10}=L_1$ **; Case B :** $u_{10}\neq L_1$

Case A: When $u_{10}=L_1$ then equations (4.11) and (4.12) become

$$u_1=u_{10}e^{-\left(a_{22}\sqrt{e_2^{\,2}+4H_2}\right)t}\ ;\quad u_2=u_{20}e^{-\left(a_{22}\sqrt{e_2^{\,2}+4H_2}\right)t} \quad (4.13);\ (4.14)$$

Here both u_1 and u_2 exponentially decay with the same Characteristic time $1/a_{22}\sqrt{e_2^{\,2}+4H_2}$, the initial values ($u_{10}$ and u_{20}) may however be different. Hence the equilibrium point is ***stable***.

The solution curves in this case are illustrated as follows

Case A.1: $u_{10}>u_{20}$

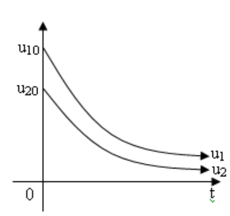


Fig 4.

Case A. 2: $u_{10}<u_{20}$

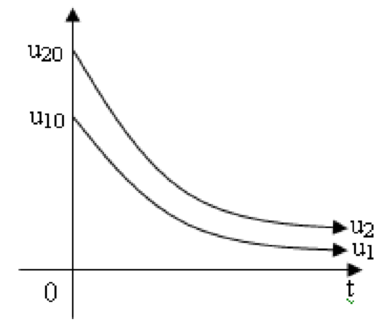


Fig 5.

Case A.1: $u_{10} = L_1$

In this case the perturbations in the commensal species always out numbers the host species in natural growth rate as well as in its initial population strength. It is noted that both the commensal and the host converge asymptotically to equilibrium point as shown in Fig.4.

Case A.2:

The perturbations in the host species dominate over the commensal species in it its initial population strength. Also both the species move towards to the equilibrium point as seen in Fig.5.

Trajectories of Perturbed Species*:*

Eliminating 't' between the equations (4.13) and (4.14), we obtain

$$\frac{u_1}{u_{10}}=\frac{u_2}{u_{20}} \quad (4.15)$$

and the corresponding trajectory is a straight line shown in Fig.6.

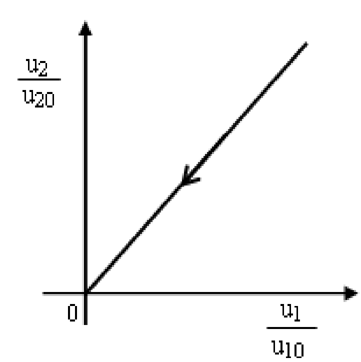
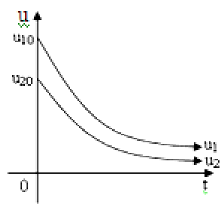
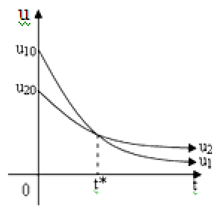
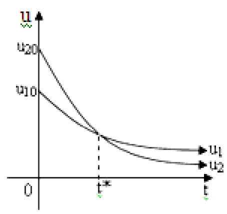
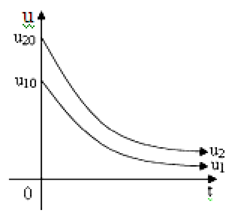


Fig 6.

Case B : $u_{10}\neq L_1$

The solutions curves of (4.11) and (4.12) are shown in Figures (7) to (10) and the remarks are presented in below.

	$a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\ 2}+4H_2}\right]\right)$ $<\ a_{22}\sqrt{e_2^{\ 2}+4H_2}$	$a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\ 2}+4H_2}\right]\right)>$ $a_{22}\sqrt{e_2^{\ 2}+4H_2}$
$u_{10}>u_{20}$	**Case B (i).**  **Fig 7.**	**Case B (ii).**  **Fig 8.**
$u_{10}<u_{20}$	**Case B. (iii).**  **Fig 9.**	**Case B. (vi).**  **Fig 10.**

OBSERVATIONS

Case B. (i): In this case the perturbation in the commensal continues to out-number the host as shown in Fig.7. However both converge asymptotically to the equilibrium point.

Case B. (ii): Initially the perturbations in the commensal out-numbers the host and this continues up to the time

$$t^*=\frac{1}{a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\ 2}+4H_2}\right]\right)-a_{22}\sqrt{e_2^{\ 2}+4H_2}}\log\left(\frac{u_{10}-L_1}{u_{20}-L_1}\right)$$

after which, the host out-numbers the commensal. It is illustrated in Fig.8.

Case B. (iii): The perturbation in Initially the host out-numbers the commensal and this continues up to the time

$$L_1^1=-L_1$$

$$t^*=\frac{1}{a_{22}\sqrt{e_2^{\ 2}+4H_2}-a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\ 2}+4H_2}\right]\right)}\log\left(\frac{u_{10}-L^1{}_1}{u_{20}-L^1{}_1}\right)$$

after which, the dominance is reversed is shown in Fig.9.

Case B. (vi): In this case the perturbation in the host continues to out-number the commensal as shown in Fig.10.

Trajectories of Perturbed Species:

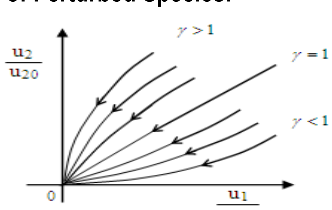


Fig 11.

Eliminating 't' between the equations (4.13) and (4.14), we obtain

$$\frac{u_1}{u_{10}}=\left(\frac{L_1}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)+\left(1-\frac{L_1}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{\gamma} \tag{4.16}$$

where $\gamma=\dfrac{a_{11}\left(k_1+\frac{c}{2}\left[-e_2+\sqrt{e_2^{\ 2}+4H_2}\right]\right)}{a_{22}\sqrt{e_2^{\ 2}+4H_2}}$ and the resulting curves are parabolic type and are shown in Fig 11. This figure exhibits the stability of the equilibrium state.

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