Journal of Experimental Sciences 2012, 3(4): 10-12
ISSN: 2218-1768
Available Online: http://jexpsciences.com/

Free A*-algebras

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Abstract
In this paper we studied the free A*-algebras, the sub A*-algebra generated by a subset and its characterization, an A*-algebra freely generated by a subset and introduced the concept of A*-field of sets. Also we established some theorems on making A*-field of sets into A*-algebras.

Keywords: A*-algebra, Free A*-algebra, A* field of sets

INTRODUCTION

In 1994, P. Koteswara Rao introduced the concept of A*-algebras and studied their equivalence with Adas, C – algebras, Stone type representation and introduced the concept of A*- clones and the if-then-else structure over A*- algebras and ideals of A*-algebras. In this paper we introduced the concept of free A*-algebras analogous to the free Boolean algebras.

PRELIMINARIES

Definition: For any non-empty set X, a class *M* of subsets of X which is closed under finite union of sets, finite intersection of sets, complementation of sets is called a field of sets.

Definition: A Boolean algebra is an algebra $(B, \vee, \wedge, (-)', 0, 1)$ with two binary operations, one unary operation(called complementation), and two nullary operations which satisfies:

(1) $(B, \vee, \wedge)$ is a distributive lattice.
(2) $x \wedge 0 = 0, \quad x \vee 1 = 1$
(3) $x \wedge x' = 0, x \vee x' = 1$

Definition: An indexed set $\{B_t\}_{t \in T}$ of subalgebras of a Boolean algebra B is said to be independent if $a_1 \wedge a_2 \wedge \ldots \wedge a_n \neq 0$ for every finite sequence of non zero elements a_i choosen from subalgebras B_t with different indices.

Definition: Let $\{B_t\}_{t \in T}$ be an indexed set of Boolean algebras. By a Boolean product of $\{B_t\}_{t \in T}$ we mean any pair $\{\{i_t\}_{t \in T}, B\}$ such that

a) B is a Boolean algebra.
b) For every $t \in T$, $i_t : B_t \to B$ is an isomorphism.
c) The indexed set $\{i_t(B_t)\}_{t \in T}$ of subalgebras of B is independent.
d) The union of all subalgebras $\{i_t(B_t)\}_{t \in T}$ generates B.

Definition: Suppose $\{X_t\}_{t \in T}$ is an indexed set of non-empty sets. Let X be the Cartesian product of X_t's. Let M_t be the field of subsets of X_t. For every set $A \subseteq X_t$, let A* be the set of all points $x \in X$ whose t^{th} coordinate $x_t \in A$ and let M_t^* be the field composed of all sets A* where $A \in M_t$. The field *M* of subsets of X generated by all the classes $\{M_t^*\}_{t \in T}$ is called the field product of $\{M_t\}_{t \in T}$.

Definition: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A* - algebra if it satisfies :
For $a, b, c \in A$

i. $a_\pi \vee (a_\pi)^{\sim} = 1$, $(a_\pi)_\pi = a_\pi$, where $a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$.
ii. $a_\pi \vee b_\pi = b_\pi \vee a_\pi$
iii. $(a_\pi \vee b_\pi) \vee c_\pi = a_\pi \vee (b_\pi \vee c_\pi)$
iv. $(a_\pi \wedge b_\pi) \vee (a_\pi \wedge (b_\pi)^{\sim}) = a_\pi$
v. $(a \wedge b)_\pi = a_\pi \wedge b_\pi$, $(a \wedge b)^\# = a^\# \vee b^\#$, where $a^\# = (a_\pi \vee a^{\sim}{}_\pi)^{\sim}$
vi. $a^{\sim}{}_\pi = (a_\pi \vee a^\#)^{\sim}$, $a^{\sim\#} = a^\#$
vii. $(a * b)_\pi = a_\pi$, $(a * b)^\# = (a_\pi)^{\sim} \wedge (b^{\sim}{}_\pi)^{\sim}$
viii. $a = b$ if and only if $a_\pi = b_\pi$, $a^\# = b^\#$. We write 0 for $1^{\sim}$, 2 for $0 * 1$.

Example: 3 $=\{0,1,2\}$ with the operations defined below is an A* - algebra

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$*$	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x_π	0	1	0
$x^\#$	0	0	1

Definition : Let $(A_1, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ and $(A_2, \wedge, (-)^{\sim}, (-)_\pi, *, 1)$ be two A*-algebras. A Mapping $f: A_1 \to A_2$ is called an A* - homomorphism if for all
$a, b \in A_1$

I. $f(a \wedge b) = f(a) \wedge f(b)$
II. $f(a \vee b) = f(a) \vee f(b)$
III. $f(a * b) = f(a) * f(b)$
IV. $f(a_\pi) = (f(a))_\pi$
V. $f(a^{\sim}) = (f(a))^{\sim}$

Received: Feb 10, 2012; Revised: March 15, 2012; Accepted: April 25, 2012.

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VI. $f(1) = 1$

VII. $f(0) = 0$.

Definition: Let C be a non-empty set. By a free A*-algebra on C we shall mean an A*-algebra F together with a mapping f: C→F such that for every A*-algebra A and every mapping g: C→A there is a unique A*-homomorphism h: F→A such that hog = f.
Now we state some routine theorems without proof.

Theorem: If (F,f) is a free A*-algebra on a non-empty set C then f is injective and Im f generates F.

Theorem: Let (F,f) be a free A*-algebra on a nom empty set C. Then (F^I, f^I) is also a free A*-algebra on C iff there is a unique A*-isomorphism j: F→ F^I such that jof = f^I.

Theorem: Let (B, ∧, (-)′, 0, 1) be a Boolean algebra. The *A*(B) = {(a, b) /a, b ∈ B, a ∧ b = 0 } becomes an A*- algebra with the following A*- algebraic operations:
For a = (x_π, $x^\#$), b = (y_π, $y^\#$) ∈ *A*(B)

I. $a \wedge b = (x_\pi y_\pi,\ x_\pi y^\# + x^\# y_\pi + x^\# y^\#)$
II. $a \vee b = (x_\pi y_\pi + x_\pi y^\# + x^\# y_\pi,\ \ x^\# y^\#)$
III. $a * b = (x_\pi,\ ((x_\pi)' y^\#)$
IV. $a^{\sim} = (x^\#, x_\pi)$
V. $a_\pi = (x_\pi, (x_\pi)')$
VI. 1 = (1, 0), 0 = (0, 1), 2 = (0, 0).

Theorem : If A is an A*- algebra then

I. *B*(*A*(B)) ≅ B.
II. *A*(*B*(A)) ≅ A.

Theorem: Let A_1,A_2 be A*- algebras and B_1,B_2 be Boolean algebras then

I. A_1 ≅ A_2 iff *B*(A_1) ≅ *B*(A_2).
II. B_1 ≅ B_2 iff *A*(B_1) ≅ *A*(B_2).

MAIN RESULTS

Theorem: For every non-empty set C, there exists a free A*-algebra on C.

Proof: Let *F* = {θ / θ: C→3 = {0, 1, 2}, θ(c) = 0 for all most all c∈ C }
Define ∨, ∧, $(-)^{\sim}$,$(-)_\pi$, ⋆, 0,1, 2 as follows:
Let θ, ξ ∈ *F.* For all c∈ C, define
(θ ∨ ξ)(c) = (θ)(c) ∨ (ξ)(c)
(θ ∧ ξ)(c) = (θ)(c) ∧ (ξ)(c)
(θ ⋆ ξ)(c) = (θ)(c) ⋆ (ξ)(c)
$\theta^{\sim}(c) = (\theta(c))^{\sim}$
0(c) = 0
1(c) = 1
2(c) = 2
Then *F* is an A*-algebra.
Now define f : C→ *F* as follows:
For any c∈ C, f(c) : C → **3** by f(c)(d) = 1 if d = c
= 0 if d ≠ c

Claim: (***F***, f) is a free A*-algebra on C.
Let A be any A*-algebra and g : C → A be a mapping.
Define h : ***F*** → A as follows:
Let θ ∈ ***F***

$$h(\theta) = \vee_{c\in C} \theta(c)\, g(c)$$

Where $\theta(c).g(c) = (\theta(c).g(c))_\pi * (\theta(c).g(c))^\#$
$(\theta(c).g(c))_\pi = (\theta(c) \wedge g(c))_\pi \vee (\theta(c)^\# \wedge g(c)^\#)$
$(\theta(c).g(c))^\# = (\theta(c)_\pi \wedge g(c)^\#) \vee (\theta(c)^\# \wedge g(c)_\pi)$
Clearly h is an A*- homomorphism
$h(f(c)) = \vee_{t\in C} (f(c))(t).\, g(t)$ = g(c)
Therefore hog = g.
Uniqueness of h:
Let θ ∈ ***F***, t∈ C
θ (t) = θ (t).1

$$= \vee_{c\in C} \theta(c).[f(c)](t)$$
$$= [\vee_{c\in C} \theta(c).f(c)](t)$$
$$Therefore\ \theta = \vee_{c\in C} \theta(c) f(c)$$

Suppose $\bar{h}: F \to A$ is another A*- homomorphism such that $\bar{h} o f = g$

$$\bar{h}(\theta) = \bar{h}(\vee_{c\in C} \theta(c) f(c))$$
$$= \vee_{c\in C} \theta(c) \overline{h}(f(c))$$
$$= \vee_{c\in C} \theta(c) g(c)$$
$$= h(\theta)$$

Therefore $\bar{h} = h$

Remerk: The free A*-algebra *F* constructed as in the above theorem is called the free A*-algebra on C.

Definition: Suppose A is an A*-algebra and $C \subseteq A$. Then A is said to be freely generated by C if

(i) <C> = A
(ii) Every mapping f : C → A^I, A^I is another A*-algebra, can be extending uniquely to a homomorphism h : A→ A^I .

Theorem: If A is an A*-algebra and B = B(A), $C \subseteq A.$ then A is freely generated by C if and only if B is freely generated by C_π, where C_π = {a_π/ a ∈ C}.

Proof: Suppose A is an A*-algebra, B = B(A) and $C \subseteq A.$ .Assume that A is freely generated by C.
We have to prove that B is freely generated by C_π = {a_π/ a ∈ C}.
Let g : C → ***B′*** is a mapping where ***B′*** is another Boolean algebra.

Define f : C → *A*(***B′***) by f(a) = ($g(a)_\pi$, $g(a)_\pi^{\sim}$).

Since C generates A, f can be extended to a unique homomorphism

h :A → *A*(*B′*).
h_π : B →B(*A*(*B′*)) is also a unique homomorphism of Boolean

algebras.

Since $B(A(B')) \cong B'$, let I : B(A(B′)) is an isomorphism such that $I(a, a^{\sim}) = a, \ \forall (a, a^{\sim}) \in B(A(B'))$.

Then $I\, h_\pi : B \to B'$ is a homomorphism and $I\, h_\pi$ is a unique extension of g.

Therefore C_π generates B freely.

Conversely assume that C_π generates B freely.

We have to prove that A is freely generated by C.

Suppose $f : C \to A'$ is another mapping where A' is another A*-algebra.

Define $f_\pi : C_\pi \to B(A')$ by $f_\pi(a) = (f(b))_\pi$

Since C_π generates B freely $\exists$ a unique homomorphism $g : B \to B(A')$

Such that $f_\pi(a) = g(a) \ \ \forall a \in C_\pi$.

Define $h : A \to A'$ by $h(a) = (g(a_\pi) * g(a_\pi))$

Then h is a homomorphism. Since g is unique, his also unique.

We now show that h = f on C.

$$\text{Let } a \in C \Rightarrow a_\pi \in C_\pi$$

$$\begin{aligned} h(a) = (g(a_\pi) * g(a_\pi)^{\sim}) &= f_\pi(a_\pi) * (f_\pi(a_\pi)^{\sim}) \\ &= [f(a)_\pi * ((f_\pi(a_\pi)^{\sim}] \\ &= f(a) \end{aligned}$$

Therefore h is a unique extension of f.

Therefore C generates A freely.

Definition: Let X be a non-empty set. A class $F^* = \{(A_1, A_2) \,/\, A_1, A_2 \subseteq X, \ A_1 \cap A_2 = \phi\}$ is called an A*-field of subsets of X if

(i) $(X, \phi) \in F^*$

(ii) $(A_1, A_2) \in F^* \Rightarrow (A_2, A_1) \in F^*$

(iii) $(A_1, A_2), (B_1, B_2) \in F^* \Rightarrow (A_1B_1, A_1B_2 + A_2B_1 + A_2B_2)$

(iv) $(A_1, A_2), (B_1, B_2) \in F^* \Rightarrow (A_1, A_1^C B_2) \in F^*$

Juxtaposition and addition stand for intersection and union of sets

From the above definition immediately we have the following theorem.

Theorem: Let X be a non-empty set. A class $F^* = \{(A_1, A_2) \,/\, A_1, A_2 \subseteq X,$ $A1 \cap A2 = \phi\}$. Then F^* is an A*-algebra with the following operations:

(i) $1 = (X, \phi), 0 = (\phi, X), 2 = (\phi, \phi)$

(ii) $(A_1, A_2)_\pi = (A_2, A_1^C)$

(iii) $(A_1, A_2)^C = (A_2, A_1)$

(iv) $(A_1, A_2) * (B_1, B_2) = (A_1, A_1^C B_2)$

(v) $(A_1, A_2) \wedge (B_1, B_2) = (A_1B_1, A_1B_2 + A_2B_1 + A_2B_2)$

(vi) $(A_1, A_2) \vee B_1, B_2) = (A_1B_1 + A_1B_2 + A_2B_1, A_2B_2)$

Proof: It is routine to verify the axioms in 1.6.

Theorem: Let F^* be an A*- field of subsets of a non-empty set X and $F = \{A \,/\, (A, A^C) \in F^*\}$

Then F is a field of subsets of X and $B(F^*) \cong F$.

Proof: Suppose F^* is an A*- field of subsets of X.

We have to prove that $F = \{A \,/\, (A, A^C) \in F^*\}$ is a field of subsets of X.

Let $A, B \in F$

Consider $(A, A^C) \wedge (B, B^C) = (AB, AB^C + A^CB + A^CB^C) \in F^*$

$= (AB, AB^C + A^C(B + B^C)) \in F^*$

$= (AB, AB^C + A^C) \in F^*$ $(AB, (A + A^C).(B^C + A^C)) \in F^*$

$= (AB, A^C + B^C) \in F^*$

$= (AB, (AB)^C) \in F^*$

$= AB \in F$

Let $A \in F$. Then $(A, A^C) \in F^* \Rightarrow (A^C, A) \in F^*$

$\Rightarrow (A^C, (A^C)^C) \in F^*$

$\Rightarrow A^C \in F$

Therefore F is a field of subsets of X and clearly $F \cong B(F^*)$

Theorem: Let F is a field of subsets of a non-empty set X. Then $A(F) = \{(A, B) \,/\, A, B \in F, A \cap B = \phi\}$ is an A*- field of subsets of X.

Proof: It is routine to verify the axioms in 2.5.

We now prove the following theorem.

Theorem: Every A*- algebra A is isomorphic to an A*- field of subsets F^* of a Stone space.

Proff: Let A be an A*- algebra.

Let B = B(A). Then there exists a Stone space H such that B $\cong$ F, where F is the field of clopen subsets of H.

By the known result A(B) $\cong$ $A(F)$.

But A $\cong$ A(B). Thus A $\cong$ $A(F)$.

ACKNOWLEDGEMENT

Bringing out a special issue in *Journal of Experimental Sciences* is a great honor to the eminent mathematician Prof.N.Ch.Pattabhi RamaCharyulu. We feel that we are very happy because our article is published in this Special issue.

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