Dispersion of a solute in a Herschel–Bulkley fluid flowing in a conduit

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Abstract
The dispersion of a solute in a Herschel-Bulkley fluid is studied by using the generalized dispersion model in both pipe and channel. With this method the entire dispersion process is described as a simple diffusion process with the effective diffusion coefficient as a function of time. The results for Newtonian fluid, power law fluid and Bingham fluid are obtained as special cases by giving appropriate values to the power law index and yield stress. The effects of power law index, yield stress on the dispersion coefficient and mean concentration have been discussed computationally and graphically. The effect of power law index and yield stress is found to reduce the dispersion coefficient. It is observed that the critical time for dispersion coefficient to reach the steady state is varying with the yield stress and power law index. It is noticed that time to assume the critical value in Newtonian case is 0.5 and in the channel case the corresponding value of time is 0.55 which are in agreement with the existed results. It is also observed that in the non- Newtonian fluids this time is less than that of Newtonian fluid case and in Bingham fluid the critical value of time in pipe flow analysis (channel flow analysis) is attained at 0.45 (0.52) while in power law fluid it is at 0.43(0.48) and in the case of Herschel-Bulkley fluid it is 0.41 (0.45).

Keywords: Generalized dispersion model, power law index, non-Newtonian fluid

INTRODUCTION

The longitudinal dispersion of a solute in a solvent flowing in a conduit (pipe/channel) has wide applications in chemical engineering, biomedical engineering, environmental sciences and physiological fluid dynamics. The indicator dilution technique which is used in physiological studies to measure the cardiac output is an example in which a quantity of a dye is introduced into the blood stream and measure its concentration at a downstream point as it flows along blood. Arterial transport phenomena are important to understand vascular diseases [1]. In particular, the transport of micro molecules and dissolved gases to and through the arterial wall is another example of convective diffusion process. Also many intravenous medications have therapeutic values at low concentration, and have toxic influence at high concentration. Therefore, it is necessary to know the rate of drugs in the blood stream of circulatory system. The theory of dispersion facilitates to understand the transport of nutrients in blood vessels and various artificial devices [2-5].

First fundamental theory on dispersion is made by Taylor [6] and considered the dispersion of a solute in a circular tube and showed that when a solute is injected into a steadily flowing solvent, the lateral molecular diffusion and the variation of velocity over the cross section would spread the solute diffusivity with effective diffusivity $D_{eff} = \dfrac{a^2 w_m^2}{48 D_m}$, where D_m is the molecular diffusivity, w_m is the mean axial velocity and a is the radius of the tube. Aris [7]

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extended the analysis of Taylor using the method of moments by including the effect of axial molecular diffusion, and calculated that the effective molecular diffusivity as $D_{eff} = D_m + \dfrac{a^2 w_m^2}{48 D_m}$ and mentioned that the validity condition $\dfrac{w_m a}{D_m} \geq 6.9$ is not required. Bournia *et al.* [8] through his experimental study found that the condition $\dfrac{4L}{a} \geq \dfrac{w_m a}{D_m}$ could be relaxed effectively but that the condition $\dfrac{w_m a}{D_m} \geq 6.9$ is quite important. These experimental data did not agree with the theoretical results of Aris [7] in the low velocity range. Bailey and Gogarty [9] obtained the numerical solution of the convective diffusion equation governing the longitudinal dispersion of a solute and verified the results with the experimental results for water displacing a dilute potassium permanganate solution. Gill and Ananthakrishnan [10] studied the effect of inlet boundary conditions on the transient approach to the asymptotic Taylor-Aris theory and this was validated by the experimental work on dispersion in horizontal pipes (Reejhsingani *et al.* [11]). Gill [12] generalized Taylor's work by giving a series expansion about the mean concentration to describe local concentration distribution. The expression obtained for effective molecular diffusion is found to reduce to Taylor's results in the limit for large Peclet number and to Aris's result for small Peclet number, where axial molecular diffusion is significant. Gill and Sankarasubramanian [13] showed that the method of series solution mentioned above provides an exact solution to the unsteady convective diffusion problem for laminar flow in a circular tube provided the coefficients in dispersion model are obtained as a suitable function of time. This model was widely known as generalized dispersion model. By considering two terms in the series solution, they observed that mean concentration profile of the solute is symmetric about a point moving with the average velocity of the fluid for all values of time. Yu in a series of

papers [14-16] using a different analysis, showed that the mean concentration profile is not symmetric at small times after the injection of the solute into the fluid.

The phenomena of dispersion in non-Newtonian fluids were studied by few researchers. Based on Taylor-Aris's fundamental theory of dispersion, Sharp [17] analyzed the dispersion in non-Newtonian fluids (Casson, Bingham, power law fluids) flowing in a conduit. He showed that the dispersion coefficient changes with yield stress for Casson and Bingham fluids and power law index in the case of power law fluids. Dash et al. [18] analyzed the dispersion of a solute in a Casson fluid in a conduit (pipe/channel) employing the generalized dispersion model of Gill and Sankarasubramanian [13]. He showed that the effect of yield stress reduced the rate of dispersion of passive species in the flow. The effective diffusivity increases with time and eventually attains its steady state value below a critical time (0.48 a^2/D_m) for dispersion in a pipe and (0.55 a^2/D_m) in channel, where a is the radius of the pipe and D_m is the molecular diffusivity.

It is reported [19] that blood obeys Casson equation only in a limited range, except at very high and very low shear rate and that there is no difference between the Casson plots and the Herschel–Bulkley plots of experimental data over the range where the Casson plot is valid. It is observed that the Casson fluid model can be used for moderate shear rates γ < 10/s in smaller diameter tubes whereas the Herschel– Bulkley fluid model can be used at still lower shear rate of flow in very narrow arteries where the yield stress is high [19, 20]. Further, the mathematical model of Herschel–Bulkley fluid also describes the behaviour of Newtonian fluid, Bingham fluid and power law fluid by taking appropriate values of the parameters viz. yield stress and power law index.

In this paper, the dispersion of a solute in a Hershel fluid flowing in a conduit (both pipe and channel) is studied using the generalized dispersion model of Gill and Sankarasubramanian [13]. The motivation for studying this problem is to understand the dispersion of dyes/drugs/nutrients in the blood stream of the cardiovascular system. The mathematical formulation for dispersion in pipe flow and corresponding solutions are presented in section 2. The channel flow analysis is presented in section 3. The effects of yield stress of the fluid and power law index on the dispersion coefficient and mean concentration are discussed in section 4 and conclusions are presented in section 5.

PIPE FLOW ANALYSIS
Mathematical Formulation

The dispersion of a solute which is initially of $\overline{z}_s$ units in length distributed in a straight circular tube of radius 'a' is considered (The schematic diagram is shown in figure 1). For axi-symmetric, fully developed, laminar flow of a Herschel-Bulkley fluid in a pipe, the unsteady convective diffusion equation which describes the local concentration $\overline{c}$ of the solute as a function of longitudinal (axial) coordinate $\overline{z}$, transverse (radial) coordinate $\overline{r}$ and time $\overline{t}$ can be written in the form

$$\frac{\partial \overline{C}}{\partial \overline{t}} + \overline{w}\frac{\partial \overline{C}}{\partial \overline{z}} = D_m\left(\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}(\overline{r}\frac{\partial}{\partial \overline{r}}) + \frac{\partial^2}{\partial \overline{z}^2}\right)\overline{C} \qquad (1)$$

where $\overline{w}$ is the axial velocity of the fluid in pipe and D_m is coefficient of molecular diffusion (molecular diffusivity) which is assumed to be constant.

For the slug input of solute length $\overline{z}_s$ under consideration, the initial and boundary conditions are given by

$$\overline{C}(0, \overline{r}, \overline{z}) = C_0 \qquad \text{if} \quad |\overline{z}| \le \overline{z}_s/2 \qquad (2a)$$
$$\overline{C}(0, \overline{r}, \overline{z}) = 0 \qquad \text{if} \quad |\overline{z}| > \overline{z}_s/2 \qquad (2b)$$
$$\overline{C}(\overline{t}, \overline{r}, \infty) = 0 \qquad (2c)$$
$$\frac{\partial \overline{C}}{\partial \overline{r}}(\overline{t}, 0, \overline{z}) = 0 = \frac{\partial \overline{C}}{\partial \overline{r}}(\overline{t}, a, \overline{z}) \qquad (2d, e)$$

Conditions (2a) and (2b) constitute the initial conditions, (2c) constitutes the far downstream condition, and (2d) and (2e) constitute respectively, the symmetry condition at the centre and the no-flux condition at impermeable wall of the pipe. We introduce the following non – dimensional variables

$$C = \frac{\overline{C}}{C_0}, \quad w = \frac{\overline{w}}{w_0}, \quad r = \frac{\overline{r}}{a}, \quad z = \frac{D_m \overline{z}}{a^2 w_0}, \quad t = \frac{D_m \overline{t}}{a^2} \qquad (3)$$

where $\quad w_0 = -\frac{a^2}{4\mu}\frac{d\overline{p}}{d\overline{z}} \qquad (4)$

is the characteristic velocity (centerline velocity in a Poiseuille flow), μ is the Newtonian viscosity (viscosity at high rate of shear) of the fluid and $\frac{d\overline{p}}{d\overline{z}}$ is the applied pressure gradient along the axis of the pipe.

With the above substitutions, the unsteady convective diffusion equation (1), in non – dimensional form, becomes

$$\frac{\partial C}{\partial t} + w\frac{\partial C}{\partial z} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{Pe^2}\frac{\partial^2}{\partial z^2}\right]C \qquad (5)$$

where $Pe = \frac{a w_0}{D_m}$, is the Peclet number $\qquad (6)$

In dimensionless form, the initial and boundary conditions (2) are reduced to

$$C(0, r, z) = 1 \qquad \text{if} \quad |z| \le z_s/2 \qquad (7a)$$
$$C(0, r, z) = 0 \qquad \text{if} \quad |z| > z_s/2 \qquad (7b)$$
$$C(t, r, \infty) = 0 \qquad (7c)$$
$$\frac{\partial C}{\partial r}(t, 0, z) = 0 = \frac{\partial C}{\partial r}(t, 1, z) \qquad (7d, e)$$

The constitutive equation for a Herschel-Bulkley fluid relating the stress (τ) and rate of strain $\left(\frac{dw}{dr}\right)$ in the non dimensional form is given by

$$\tau = \tau_y + (-\frac{dw}{dr})^{1/n} \qquad \text{if} \quad \tau \ge \tau_y \qquad (8a)$$

and $\quad \frac{dw}{dr} = 0, \qquad \text{if} \quad \tau \le \tau_y \qquad (8b)$

where $\quad \tau = \frac{\overline{\tau}}{2\mu(w_0/a)} \qquad (8c)$

and $\quad \tau_y = \frac{\overline{\tau}_y}{2\mu(w_0/a)} \qquad (8d)$

where 'n' is the power law index, τ and τ_y are the non-dimensional shear stress and yield stress respectively. The above

relations correspond to vanishing of velocity gradient in the region where the shear stress τ is less than the yield stress τ_y which implies a plug flow whenever. $\tau \leq \tau_y$ This model describes the Newtonian fluid if $\tau_y = 0$ and $n = 1$, a power law fluid model by taking $\tau_y = 0$ and $n \neq 1$ and a Bingham plastic fluid by taking $\tau_y \neq 0$ and $n = 1$.

The velocity distribution for axi-symmetric, fully developed, steady, laminar flow of a Herschel-Bulkley fluid in a pipe by solving the momentum equation under no-slip boundary conditions, in non-dimensional form is obtained as

$$w = w_+ = \frac{2}{n+1}\left\{(1-r_p)^{n+1}-(r-r_p)^{n+1}\right\} \quad \text{if} \quad r_p \leq r \leq 1 \quad (9a)$$

$$w = w_- = w_p = \frac{2}{n+1}(1-r_p)^{n+1} \quad \text{if} \quad 0 \leq r \leq r_p \quad (9b)$$

where $r_p = \dfrac{a\,\overline{\tau}_H}{2\mu\,w_o} = \tau_y$ (10)

is the dimensionless radius of plug flow region. The subscripts '+' and '-' correspond to the values for the shear flow ($r_p \leq r \leq 1$) and plug flow ($0 \leq r \leq r_p$) regions respectively. The fluid particles in the plug flow region, do not move by themselves, but are merely carried along by the fluid particles in the adjacent shear flow region as a solid body with a constant velocity w_p which is the plug flow velocity.

The mean velocity of the fluid in dimensionless form is given by

$$w_m = \frac{2}{n+1}\left\{(1-r_p)^{n+1}-\frac{2(1-r_p)^{n+1}}{n+2}+\frac{2(1-r_p)^{n+3}}{(n+2)(n+3)}\right\} \quad (11)$$

Method of Solution

We consider the convection across a plane moving with an average velocity w_m of the fluid. Hence, we define a new coordinate system moving with new axial coordinate z_1, given by

$$z_1 = z - w_m t. \quad (12)$$

The solution of equation (5) along with the conditions (7) is formulated as a series expansion in $\dfrac{\partial^j C_m}{\partial z^j}$, following Gill and Sankarasubramanian [13], is given by

$$C = C_m + \sum_{j=1}^{\infty} f_j(t,r)\frac{\partial^j C_m}{\partial z^j} \quad (13)$$

where $C_m = 2\displaystyle\int_0^1 C\,r\,dr$ (14)

is the mean concentration over a cross section. On transforming the unsteady convective diffusion equation (5) into the moving co-ordinate system (r, z_1, t) where z_1 is given in equation (12) and substituting equation (13) into the transformed unsteady convective equation, we get

$$\frac{\partial C_m}{\partial t}+(w-w_m)\frac{\partial C_m}{\partial z_1}-\frac{1}{Pe^2}\frac{\partial^2 C_m}{\partial z_1^2}+\sum_{j=1}^{\infty}\left[\left(\frac{\partial f_j}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_j}{\partial r}\right)\right)\frac{\partial^j C_m}{\partial z_1^j}+(w-w_m)f_j\frac{\partial^{j+1}C_m}{\partial z_1^{j+1}}\right.$$
$$\left.-\frac{1}{Pe^2}f_j\frac{\partial^{j+2}C_m}{\partial z_1^{j+2}}+f_j\frac{\partial^{j+1}C_m}{\partial t\,\partial z_1^j}\right]=0 \quad (15)$$

Assuming that the process of distributing C_m is diffusive in

nature from the time 'zero', then as in Gill and Sankarasubramanian [13] the generalized dispersion model for C_m can be written as

$$\frac{\partial C_m}{\partial t}=\sum_{i=1}^{\infty}K_i(t)\frac{\partial^i C_m}{\partial z_1^i} \quad (16)$$

with dispersion coefficient K_i as suitable functions of time t. The first two terms in the RHS of equation (16) describe the transport of C_m in axial direction z_1 through convection and diffusion respectively, and therefore, the coefficients K_1 and K_2 are termed as the longitudinal convection and diffusion coefficients for C_m. Substituting equation (16) in equation (15) and rearranging the terms, we obtain

$$\left[\frac{\partial f_1}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_1}{\partial r}\right)+(w-w_m)+K_1(t)\right]\frac{\partial C_m}{\partial z_1}+\left[\frac{\partial f_2}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_2}{\partial r}\right)+(w-w_m)f_1\right.$$
$$\left.+K_1(t)f_1+K_2(t)-\frac{1}{pe^2}\right]\frac{\partial^2 C_m}{\partial z_1^2}+\sum_{j=1}^{\infty}\left\{\frac{\partial f_{j+2}}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_{j+2}}{\partial r}\right)+(w-w_m)f_{j+1}(t,r)\right.$$
$$\left.-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_{j+2}}{\partial r}\right)+(w-w_m)f_{j+1}(t,r)-\frac{1}{Pe^2}f_j(t,r)+\sum_{i=1}^{j-1}f_{j+2-i}(t,r)\,K_i(t)+K_{j+2}(t)\right\}\frac{\partial^{j+2}C_m}{\partial z_1^{j+2}}=0 \quad (17)$$

Comparing the coefficient of $\dfrac{\partial^j C_m}{\partial z_1^j}$, $j=1,2$ .., we get an infinite set of differential equations given by

$$K_1(t)+\frac{\partial f_1}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_1}{\partial r}\right)+(w-w_m)=0 \quad (18a)$$

$$K_2(t)-\frac{1}{Pe^2}+\frac{\partial f_2}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_2}{\partial r}\right)+[(w-w_m)+K_1(t)]f_1=0 \quad (18b)$$

$$K_{j+2}(t)+\frac{\partial f_{j+2}}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_{j+2}}{\partial r}\right)+(w-w_m)f_{j+1}-\frac{1}{Pe^2}f_j+\sum_{i=1}^{j+1}K_i(t)f_{j+2-i}=0 \quad (18c)$$

for $j=1, 2\ldots$ with $f_0 = 1$.

From equation (7) and (13), we get the initial and boundary conditions on f_j's as

$$f_j(0,r)=0 \qquad\qquad j=1, 2\ldots. \quad (19a)$$

$$\frac{\partial f_j}{\partial r}(t,0)=0=\frac{\partial f_j}{\partial r}(t,1) \qquad j=1, 2,\ldots.. \quad (19\,b,c)$$

and from equation (13) and (14), we have

$$\int_0^1 f_j\,r\,dr=0 \qquad\qquad j=1, 2,\ldots \quad (20)$$

Multiplying equations (18a) (18b) and (18c) by r and integrating from 0 to 1, using the condition (20), we have

$$K_1(t)=-2\int_0^1(w-w_m)\,r\,dr=0 \quad (21)$$

$$K_2(t)=\frac{1}{Pe^2}-2\int_0^1 f_1(t,r)\,w(r)\,r\,dr \quad (22)$$

and $K_{j+2}(t)=-2\displaystyle\int_0^1 f_{j+1}(t,r)\,w(r)\,r\,dr$, $j=1,2,\ldots\ldots$ (23)

The function f_1 is the most important coefficient of the series in equation (13). It gives the measure of deviation of the local concentration C from the mean concentration C_m. The solution to the non-homogeneous parabolic partial differential equation (18a) and

conditions (19) can be written in the form

$$f_1(t, r) = f_{1s}(r) + f_{1t}(t, r) \tag{24}$$

where $f_{1s}(r)$ is the large time solution which corresponds to Taylor-Aris's dispersion theory and f_{1t} is the transient part which describes the time-dependent nature of the dispersion phenomena corresponding to a Herschel-Bulkley fluid.
Using equations (19), (21) and (24) in (18a) we get

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f_{1s}}{\partial r}) = -(w - w_m) \tag{25}$$

$$\frac{\partial f_{1t}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f_{1t}}{\partial r}) \tag{26}$$

with initial and boundary conditions

$$\frac{df_{1s}}{dr}(r=0) = 0 = \frac{df_{1s}}{dr}(r=1) \tag{27a, b}$$

$$\frac{\partial f_{1t}}{\partial r}(t,0) = 0 = \frac{df_{1t}}{dr}(t,1) \tag{27c,d}$$

$$f_{1t}(0, r) = - f_{1s}(r) \tag{27e}$$

from (20), we have

$$\int_0^1 f_{1t}\, r\, dr = = -\int_0^1 f_{1s}\, r\, dr = 0 \tag{28}$$

The solution for f_{1s} is , obtained from equation (25) subject to the conditions (27) and (28), is given by

$$f_{1s}(r) = f_{1s^+}(r) = \frac{2}{n+1}$$

$$\left\{ -\frac{(r - r_p)^{n+3}}{(n+2)\ (n+3)} + \frac{1}{(n+2)\ (n+3)}\int\frac{(r-r_p)^{n+3}}{r}dr \right.$$

$$\left. +\frac{(1-r_p)^{n+3}}{(n+2)}\frac{r^2}{2} - \frac{(r-r_p)^{n+3}}{(n+2)\ (n+3)}\frac{r^2}{2} \right\} + B_1 \text{ if } r_p \le r \le 1 \tag{29a}$$

$$f_{1s}(r) = f_{1s^-}(r) = \frac{2}{n+1}\left\{\frac{(1-r_p)^{n+2}}{(n+2)} - \frac{(1-r_p)^{n+3}}{(n+2)(n+3)}\right\}\frac{r^2}{2} + B_2 \text{ if } 0 \le r \le r_p \tag{29b}$$

where

$$B_1 = -\frac{2^2}{n+1}\left\{\frac{1}{(n+2)(n+3)}\left(\frac{(1-r_p)^{n+5}}{(n+4)(n+5)} - \frac{(1-r_p)^{n+4}}{(n+4)}\right) + \frac{(1-r_p)^{n+2}}{8\,(n+2)} - \frac{(1-r_p)^{n+3}}{8\,(n+2)\,(n+3)}\right.$$

$$\left. +\frac{1}{(n+2)\,(n+3)}\int_{r_p}^1 r\left(\int_r\frac{(r-r_p)^{n+3}}{r}dr\right)dr + \frac{r_p^2}{2\,(n+2)\,(n+3)}\int_{r=r_p}\frac{(r-r_p)^{n+3}}{r}dr\right\} \tag{29c}$$

$$B_2 = \frac{2}{(n+1)\ (n+2)\ (n+3)}\int_{r=r_p}\frac{(r-r_p)^{n+3}}{r}dr + B_1. \tag{29d}$$

The integrals in (29 a, c, d) are evaluated numerically.
The solution for f_{1t} is obtained from the equation(26) subject to the conditions (27) and (28) which is given by

$$f_{1t} = \sum_{m=1}^{\infty} A_m J_0(\lambda_m r)\, e^{-\lambda_n^2 t} \tag{30}$$

where

$$A_m = -\frac{\int_0^1 f_{1s}\, J_0(\lambda_m r)\, r\, dr}{\int_0^1 r\left(J_0^2(\lambda_m r)\right)^2 dr} \tag{31a}$$

and λ_m's are the solutions of the equation $J_1(x) = 0$. (31b)
Here J_0, and J_1 are the Bessel functions of first kind of order zero and one respectively.
In the generalized dispersion model given by equation (16), K_2 plays an important role. From the equation (22), we can see that it depends on the function f_{1t}. We can solve the equation (22) for K_2 by substituting the expression for f_{1s} and f_{1t}. Once $K_2(t)$ is known, then $f_2(t, r)$ can be obtained from equation (18b) in a similar manner to that $f_1(t, r)$. In this way we can find $K_3(t)$, $f_3(t, r)$, $K_4(t)$, $f_4(t, r)$ .... and so on. Since the expression for $f_1(t, r)$ and $K_2(t)$ are complex in nature, it is very difficult to evaluate $f_2(t, r)$, $K_3(t)$, ... and so on. But for dispersion in a Newtonian fluid which corresponds to $r_p = \tau_y = 0$, it was shown that [13], $K_3(t \to \infty) = -1/23040$ and the magnitude of higher order coefficients decrease further. Owing to the yield stress and power law index these coefficients may decrease further in magnitude and hence they are not evaluated. Neglecting $K_3(t)$ and higher order coefficients, the generalized dispersion model leads to

$$\frac{\partial C_m}{\partial t} = K_2(t)\frac{\partial^2 C_m}{\partial z_1^2} \tag{32}$$

The initial and boundary conditions for C_m are given by

$$C_m(0, z_1) = 1 \quad if \quad |z_1| \le \frac{z_s}{2} \tag{33a}$$

$$C_m(0, z_1) = 0 \quad if \quad |z_1| > \frac{z_s}{2} \tag{33b}$$

$$C_m(t,\infty) = 0 \tag{33c}$$

The solution of the equation (32) for mean concentration with the help of the conditions (33) is given by

$$C_m = \frac{1}{2}\left[erf\left(\frac{\frac{1}{2}z_s - z_1}{2\sqrt{\xi}}\right) + erf\left(\frac{\frac{1}{2}z_s + z_1}{2\sqrt{\xi}}\right)\right] \tag{34}$$

where $\xi = \int_0^t K_2(t)\, dt$. $\tag{35}$

By using the expression for C_m and f_1 in equation (13) and neglecting the higher order terms, we can obtain the local concentration C to a first approximation as

$$C(r, z_1, t) = C_m(z_1, t) + f_1(r, t)\frac{\partial C_m}{\partial z_1}(z_1, t). \tag{36}$$

CHANNEL FLOW ANALYSIS

We use the Cartesian co-ordinate system $(\bar{x}, \bar{z})$, where $\bar{x}$ denotes the transverse co-ordinate and $\bar{z}$ denotes the axial co-ordinate, to describe the dispersion of a solute in a Herschel-Bulkley

fluid flowing in a channel. The flow is assumed to be axi-symmetric, fully developed, steady and laminar. Proceeding same as in the case of pipe flow, the unsteady convective diffusion equation for the dispersion of the solute in the channel flow in non-dimensional form can be written as

$$\frac{\partial C}{\partial t} + w(x)\frac{\partial C}{\partial z} = \left[\frac{\partial^2}{\partial x^2} + \frac{1}{Pe^2}\frac{\partial^2}{\partial z^2}\right]C \qquad (37)$$

and $Pe = \dfrac{a\,w_0}{D_m}$, is the Peclet number,

with the initial and boundary conditions, in dimensionless form as

$$C(0, x, z) = 1 \qquad if\ |z| \le z_s/2 \qquad (38a)$$
$$C(0, x, z) = 0 \qquad if\ |z| > z_s/2 \qquad (38b)$$
$$C(t, x, \infty) = 0 \qquad (38c)$$
$$\frac{\partial C}{\partial x}(t, 0, z) = 0 = \frac{\partial C}{\partial x}(t, 1, z) \qquad (38\ d,e)$$

where a is half of the channel width, and w_0 is the characteristic velocity which is given by

$$w_0 = -\frac{a^2}{4\mu}\frac{d\bar{p}}{d\bar{z}}. \qquad (39)$$

For axi-symmetric, fully developed, steady, laminar flow of a Herschel-Bulkley fluid in a channel, the velocity in non- dimensional form, is obtained as

$$w(x) = w_+ = \frac{2}{n+1}\left\{(1-x_p)^{n+1} - (x-x_p)^{n+1}\right\} if\ x_p \le x \le 1 \qquad (40a)$$

$$w(x) = w_- = w_p = \frac{2}{n+1}(1-x_p)^{n+1} \quad if\ 0 \le x \le x_p \qquad (40b)$$

where $x_p = \dfrac{a\,\bar{\tau}_y}{2\mu\,w_o} = \tau_y$ \qquad (41)

denotes half of dimensionless thickness of the plug flow region in a channel which is equal to the dimensionless yield stress τ_y of the fluid in the channel. The mean velocity, in dimensional form is given by

$$W_m = \frac{2}{n+1}\left\{(1-x_p)^{n+1} - \frac{(1-x_p)^{n+2}}{n+2}\right\} \qquad (42)$$

The unsteady convective diffusion equation (37) has to be solved for local concentration C subject to the conditions (38) with axial velocity w given in equation (40). In a similar manner to that of pipe flow analysis, we proceed to find the solution for the problem in a co-ordinate system moving with the average velocity w_m of the fluid. In the channel case the mean concentration C_m is defined as

$$C_m = \int_0^1 C\,dx \qquad (43)$$

There will be minor modifications in the solvability condition (20) due to change in the definition of C_m and expressions for K_j, j = 1, 2, ……. . we can find the solution for f_{1s}, f_{1t} and K_2 similar as in

pipe case.

The steady state solution f_{1s} is obtained as

$$f_{1s}(r) = f_{1s+}(r) = \frac{2}{n+1}\left\{\frac{(1-x_p)^{n+2}}{(n+2)}\frac{x^2}{2} - \frac{(x-x_p)^{n+3}}{(n+2)\ (n+3)}\right\} +$$

E, if $\quad r_p \le r \le 1$ \hfill (44a)

$$f_{1s}(r) = f_{1s-}(r) = \frac{2}{n+1}\left\{\frac{(1-x_p)^{n+2}}{(n+2)}\frac{x^2}{2}\right\} + E,\ if\ 0 \le r \le r_p \qquad (44b)$$

where $E = \dfrac{2}{(n+1)\ (n+2)}\left\{\dfrac{(1-x_p)^{n+4}}{(n+3)(n+4)} - \dfrac{(1-x_p)^{n+2}}{6}\right\}$ \qquad (44c)

The solution for f_{1t} is given by

$$f_{1t} = \sum_{m=1}^{\infty} A_m\ e^{-\lambda_m^2 t}\cos(\lambda_m x) \qquad (45a)$$

where $\quad A_m = -\dfrac{\displaystyle\int_0^1 f_{1s}\cos(\lambda_m x)\,dx}{\displaystyle\int_0^1 \cos^2(\lambda_m x)\,dx}$ \hfill (45b)

and $\quad \lambda_m = m\,\pi$, m = 1, 2, ………… \hfill (45c)

RESULTS AND DISCUSSION

The aim of the present study is to investigate the dispersion of a solute in a Herschel-Bulkley fluid flowing in a conduit (pipe/channel). This analysis is significant to understand the dispersion of nutrients and drugs in circulatory system and to measure the cardiac output using Indicator Dilution Technique. The unsteady convective dispersion phenomenon is analyzed using the generalized dispersion model suggested by Gill and Sankarasubramanian [13]. Thus the entire dispersion process is described in terms of a simple diffusion process with apparent diffusion coefficient K_2 as a function of time. We obtained the results for Newtonian fluid, power law fluid and Bingham plastic fluid as special cases.

The time dependent nature of the dispersion coefficient $(K_2 - 1/Pe^2)$ with a variation in yield stress for dispersion in pipe flow is presented in fig. 2(a) and the plot of $(K_2 - 1/Pe^2)$ with time for different yield values of x_p (half thickness of the plug flow region) in channel is illustrated in fig. 2(b). The dispersion coefficient $(K_2 - 1/Pe^2)$ is observed to assume a constant value for large values of time. The time taken to reach the steady state is observed to be dependent on the plug flow parameter r_p or x_p equivalently the yield stress τ_y of the fluid). It is noticed that this critical value reaches faster in both pipe and channel flow cases as yield stress increases. However, the time taken to reach this critical value in channel flow analysis is more than that of pipe flow analysis. The dispersion coefficient $(K_2 - 1/Pe^2)$ versus time in different fluids is shown in fig. 2(c) and 2(d) for pipe and channel cases respectively. It is noticed that time taken for $(K_2 - 1/Pe^2)$ to assume the critical value in Newtonian case is 0.5 which is in agreement with the observation of Gill and Sankarasubramanian [13]. In the channel case the corresponding value of time is 0.55 in Newtonian fluid. In the case of non- Newtonian fluids this time is less than that that of Newtonian fluid case. This critical time in Casson fluid is found to be less than that of Herschel Bulkley fluid [18]. In Bingham fluid the critical value in pipe flow analysis (channel flow analysis) is attained

at t = 0.45 (0.52) while in power law fluid it is at 0.43 (0.48) and in the case of Herschel-Bulkley fluid it is at 0.41 (0.45). The dispersion coefficient $(K_2 - 1/Pe^2)$ in all the non-Newtonian fluids in pipe/channel is reduced when compared with Newtonian fluid. This reduction may be attributed to the reduction in flow rate in the non-Newtonian fluids in pipe and channels due to the non Newtonian rheology. At t = 0.3 the value of $(K_2 - 1/Pe^2)$ is 0.004105 for a Newtonian fluid and 0.002817 for a Bingham fluid. In a power law fluid there is a two fold reduction to that of the corresponding value in Newtonian case while there is a four fold reduction in the Herschel-Bulkley fluid. An exact similar trend is noticed in the case of channel flow analysis.

The time evaluation of the function f_1 for dispersion in pipe and channel flows with different values of yield stress is plotted in figs. 3 (a, b). f_1 provides a measure of deviation in the local concentration C from the mean concentration C_m. At time t = 0, f_1 is uniformly zero over the entire cross-section of the pipe/channel. It is observed that f_1 attains its steady state value f_{1s} (dashed line in figs. 3(a, b)) as t increases (which is also noticed in fig. 4 (a, b)) for for various values of yield stress. In the absence of yield stress corresponding to the Newtonian fluid, the magnitude of f_1 has maximum which is in agreement with the observation of Gill and Sankarasubramanian [13]. The presence of yield stress and increase in it (r_P or x_P) decreases f_{1s}. It is noticed from figs. 3 and 4 that the functions f_1 and f_{1s} pass through a common point for all times and for all values of yield stress (plug parameter). At this point f_1 and f_{1s} are zero and the local concentration C of the solute became equal to the mean concentration C_m. Hence, this point shall be treated as the center of mass of the solute over a cross section of the pipe/channel. This point i.e. the center of mass of the solute is found to be independent of time and yield stress r_P or x_P. The center of mass of the solute in the pipe flow analysis occurs nearer to the pipe wall while it is in the midway of the channel width. Fig. 3(c, d) and fig. 4(c, d) represents the dispersion function f_1 and steady state dispersion function f_{1s} for different fluids in pipe and channel cases respectively.

Fig. 5(a, b) describes the variation of the mean concentration with time when n = 2, r_P = 0.1 (x_P = 0.1) and z = 0.5. It is observed that the peak values of mean concentration C_m occurs at t =1.62 for different slug input lengths. The peak values of C_m in pipe analysis increase with increase in slug input length. There is a two fold increase when z_s increases from 0.004 to 0.008. A five fold increase is noticed when z_s = 0.019. Exactly a similar trend is observed for dispersion in channel flow also. However, the magnitude of the peak values of C_m is less in channel. Fig. 5(c, d) describes the mean

concentration for different fluids. It is observed that the non-Newtonian rheology of the fluid increases the value of C_m and the time taken to attain this peak value of C_m also increases in both pipe and channel analyses. In pipe (channel) flow, the peak value in the case of Newtonian fluid occurs at t = 1(0.98) for Bingham fluid (n =1, r_P = 0.1) at t = 1.15 (1.12). In the case of Power law fluid (n = 2, r_P =0) the peak value is reached at t = 1.25 (1.24) and in Herschel-Bulkley fluid (n = 2, r_P = 0.1), it is at 1.6 (1.58). The mean concentration versus the axial distance z at t = 0.03 is illustrated. in fig. 6 (a, b). The peak value of concentration occurs at z = 0.01 for all slug input lengths. The peak value changes from 0.1942 to 0.7572 when z_s changes its value from 0.004 to 0.019. In channel flow analysis the mean concentration profiles are bell shaped curves and the peak value occurs at z = 0.01. The peak value varies from 0.4844 to 0.9980 when z_s varies from 0.004 to 0.019. Fig 6 (c, d) shows C_m versus z for different fluids. It is noticed that the peak value of C_m is attained closer to the origin in the case of non-Newtonian fluids. In Herschel-Bulkley fluid the point of maximum occurs at z = 0.09, for power law fluids it is at z = 0.0112 and for Bingham plastics it is at z = 0.013. For dispersion in Newtonian fluid, the peak value is attained at z = 0.015. In channel flow analysis also the peak values of mean concentration are attained at the same locations.

CONCLUSIONS

The present investigation brings out some interesting results on the dispersion process in flows of blood modeled as a Herschel-Bulkley fluid. The convective dispersion process is analyzed employing the dispersion model of Gill and Sankarasubramanian [13]. It is observed that the dispersion coefficient which describes the entire diffusion phenomenon is changed due to the non-Newtonian behaviour of the fluid (i.e. yield stress and/or power law index). The dispersion coefficient attains a steady value for larger times. This critical time is observed to be more in Newtonian fluid than in Herschel-Bulkley fluid. This Critical value of time is found to be further smaller in Casson fluid. The rate of dispersion of the passive solute is reduced in the Herschel-Bulkley fluid. The location of the center of mass of the solute over a cross-section is found to be independent of time and yield stress. It is observed that the effect of non-Newtonian rheology on mean concentration is to increase the time to reach its peak value.
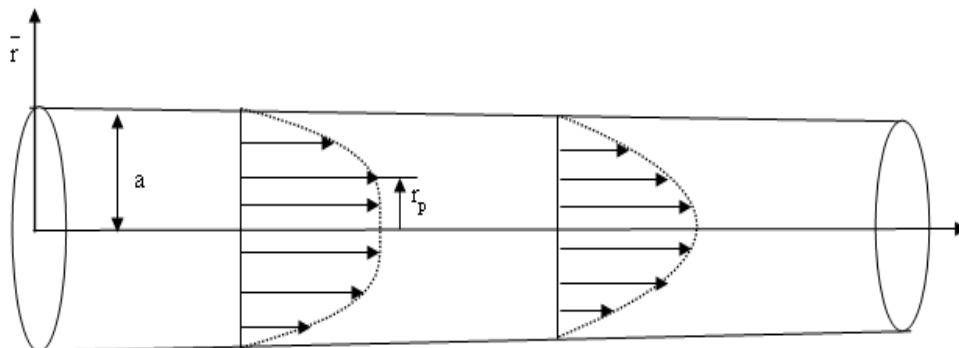

Fig 1. Schematic diagram of velocity profile for Herschel-Bulkley Fluid and Newtonian fluids in a circular tube
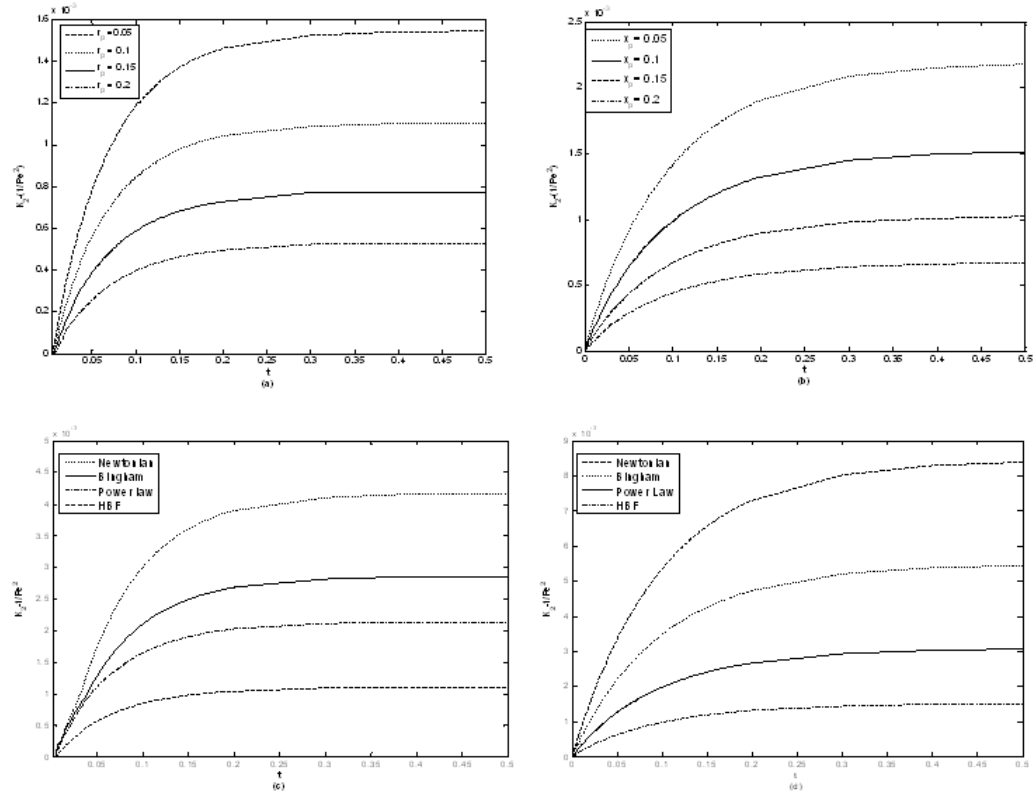
Fig 2. Variation of dispersion coefficient $K_2 - 1/Pe^2$ verses time t for different values of (a) r_P in a pipe when $n = 2$ (b) x_P in a channel when $n = 2$

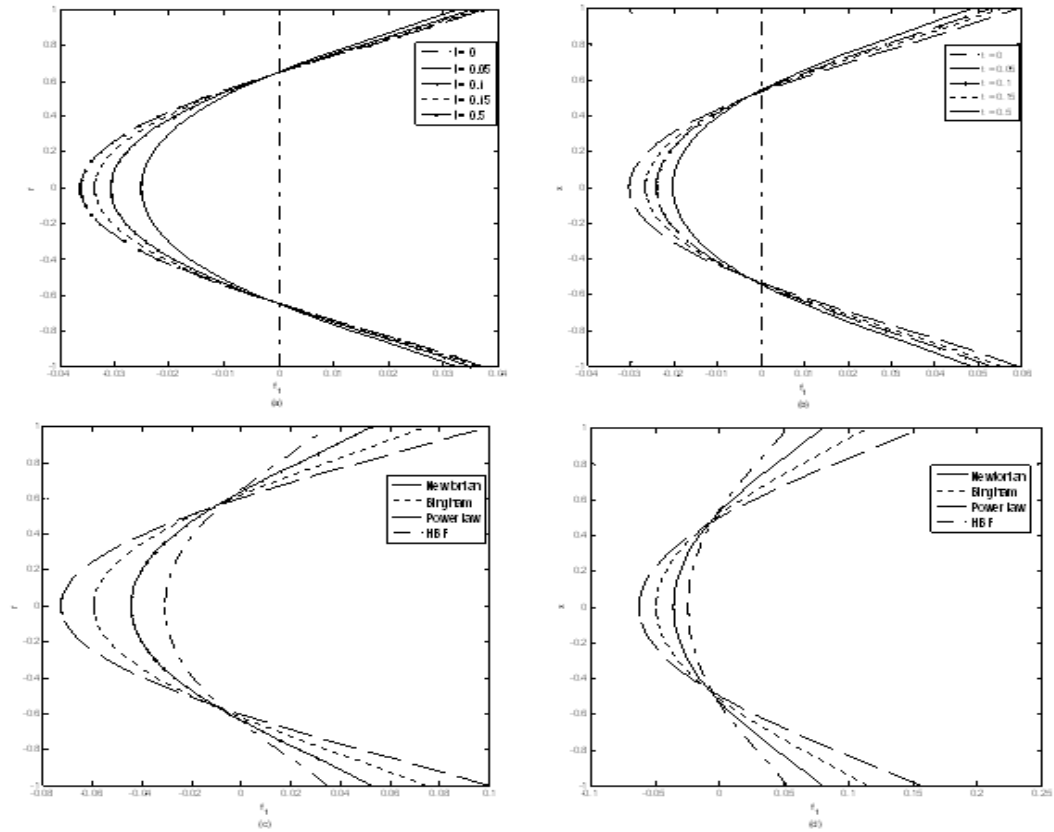(c) in pipe when $r_P = 0.1$ when $n = 2$ (d) in channel when $x_P = 0.1$ when $n = 2$

Fig 3.Distribution of dispersion function f_1 for different values of time t (a) in a pipe when $r_P = 0.1$, $n = 2$

(b) in a channel when $x_P = 0.1$, $n = 2$ (c) for different fluids in pipe   (d) for different fluids in channel
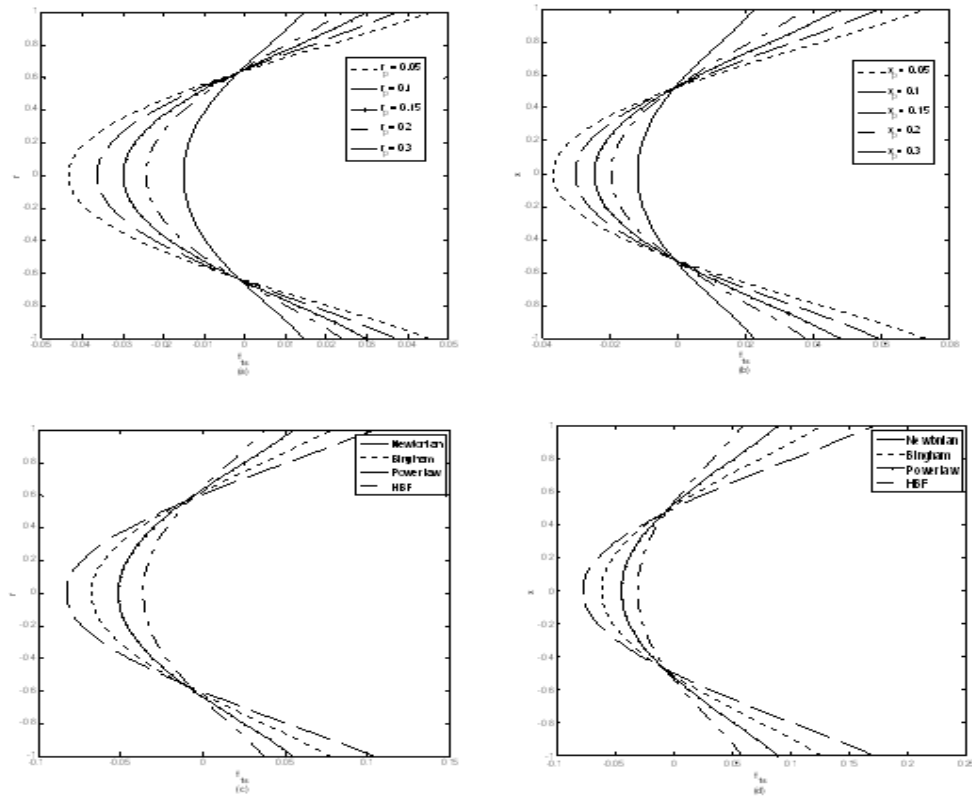
Fig 4. Distribution of steady state dispersion function f_{1s} (a) in pipe   with r_p when $n = 2$ (b) in channel with x_p when $n = 2$
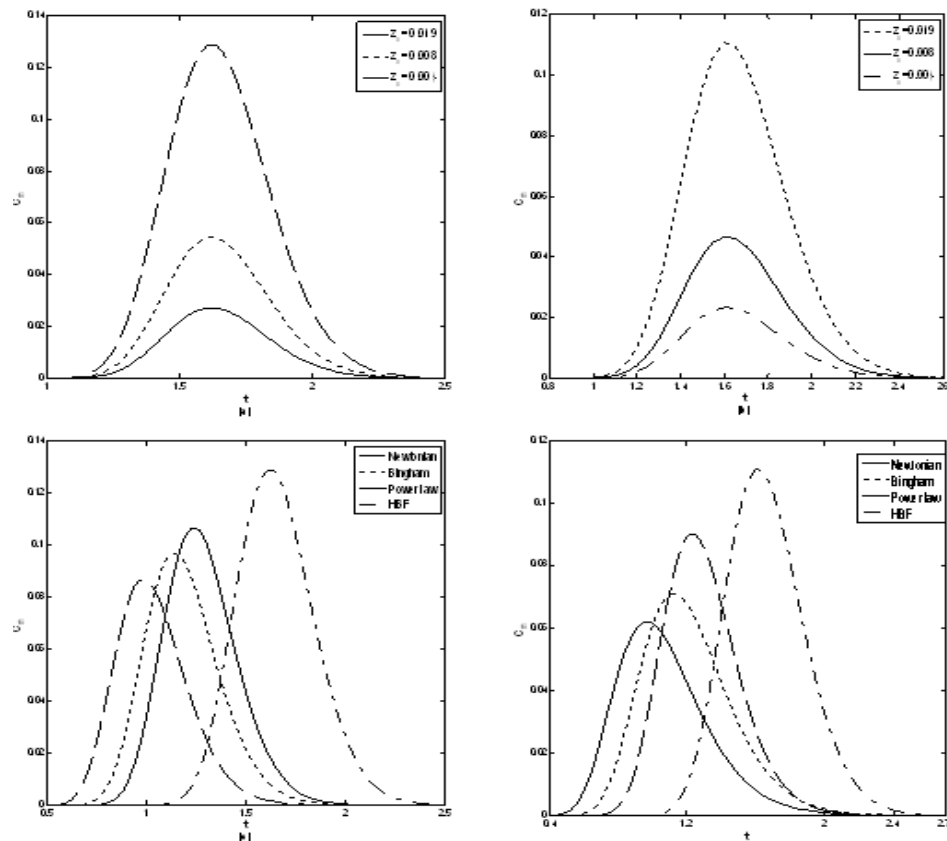(c) for different fluids in pipe (d) for different fluids in channel

Fig 5. Variation of the mean concentration C_m with t (a) for different z_s when $z = 0.5$, $r_p = 0.1$, $n = 2$ (b) for different z_s when $z = 0.5$, $x_p = 0.1$, $n = 2$
(c) for different fluids in pipe when $z = 0.5$, $z_s = 0.019$ (d) for different fluids in channel when $z = 0.5$, $z_s = 0.019$
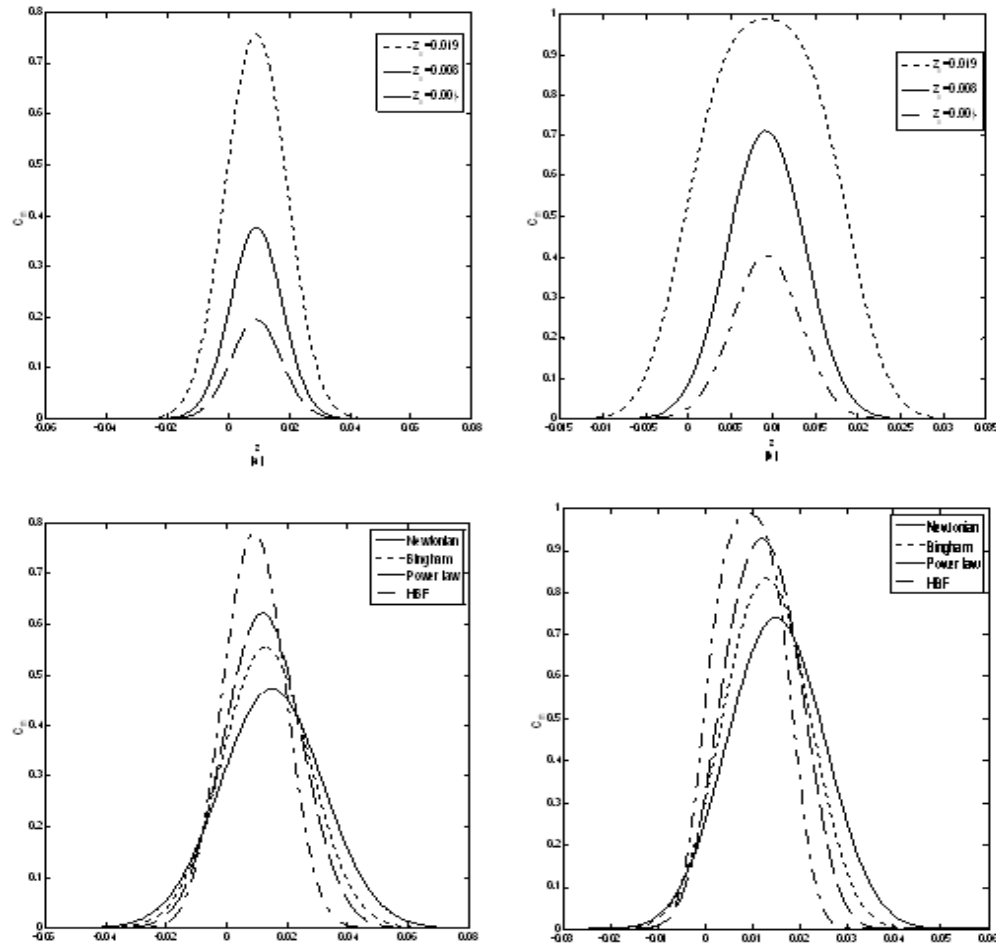
Fig 6.Variation of the mean concentration C_m with axial distance z (a) for different z_s when $t = 0.03$, $r_p = 0.1$, $n = 2$ (b) for different z_s when $t = 0.03$, $x_p = 0.1$, $n = 2$
(c) for different fluids in pipe when $t = 0.03$, $z_s = 0.019$ (d) for different fluids in channel when $t = 0.03$, $z_s = 0.019$

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