Journal of Experimental Sciences 2012, 3(2): 07-13
ISSN: 2218-1768
Available Online: http://jexpsciences.com/

journal of
experimental sciences

On the stability of a four species syn eco-system with commensal prey-predator pair with prey-predator pair of hosts-III (2^{nd} level prey-predator washed out states)

B. Hari Prasad

Asst. Prof. Dept. of Mathematics, Chaitanya Degree College (Autonomous), Hanamkonda-506001, India

Abstract

The present paper is devoted to an investigation on a Four Species (S_1, S_2, S_3, S_4) Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts (Both the Hosts are washed out states). The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are Commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 is Prey for S_4 and S_4 is Predator for S_3. The pair (S_1, S_2) may be referred as 1^{st} level Prey-Predator and the pair (S_3, S_4) the 2^{nd} level Prey-Predator. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of three of these sixteen equilibrium points: 2^{nd} Level Prey-Predator Washed Out States are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories are illustrated.

Keywords: Commensal, Eco-System, Equillibrium point, Host, Prey, Predator.

INTRODUCTION

Research in the area of theoretical Ecology was initiated in 1925 by Lotka [1] and in 1931 by Volterra [2]. Since then many Mathematicians and Ecologists contributed to the growth of this area of knowledge reported in the treatises of May [3], Smith [4], Kushing [5], Kapur [6] etc. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so on. Srinivas [7] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later Lakshminarayan and Pattabhi Ramacharyulu [8] studied Prey-Predator ecological models with partial cover for the Prey and alternate food for the Predator. Recently, Archana Reddy [9] and Bhaskara Rama Sharma [10] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Further Phani Kumar and Pattabhi Ramacharyulu [11] studied Three Species Ecosystem Consisting of a Prey, Predator and a Host Commensal to the Prey. The present authors Hari Prasad and Pattabhi Ramacharyulu [12-17] discussed on the stability of a four species Syn Eco-System.

Predation: Predation is a relationship between two species where one species kills and devours other for food. The species which kills other is called a predator and the species which is killed is called a prey. For example, a tiger that hunts is called “Predator” and a deer that is being hunted is known as the “Prey”.

Received: Dec 08, 2011; Revised: Jan 20, 2012; Accepted: Feb 15, 2012.

*Corresponding Author
B. Hari Prasad
Asst. Prof. Dept. of Mathematics, Chaitanya Degree College (Autonomous)
Hanamkonda-506001, India

Tel: +91-9399940624
Email: sumathi_prasad73@yahoo.com

Commensalism: In commensalisms one organism benefits the other without getting effected due to the interaction (i.e. it is neither benefited nor harmed). The beneficial species is the Commensal while the other benefiting species the Host. A common example is an animal using a tree for shelter-tree (Host) does not get any benefit from the animal (Commensal).

Some real-life examples of a Syn-Eco-System with Commensal Prey-Predator pair with Pray-Predator pair of Hosts are given in the following Table-1.

Table 1.

Sl. No.	Examples of S_1	Examples of S_2	Examples of S_3	Examples of S_4
1	Infusoria	Sea anemone	Arthopods	Clown fish
2	Small beetle	Remora	Fish (or) small aquatic vertebrate	Shark
3	Rabit	Golden Jackal	Deer	Tiger
4	Insects	Army Ants	Earth worms	Birds
5	Grass	Cow	Insects	Cattle egrets

A Schematic Sketch of the system under investigation is shown here under Fig 1.

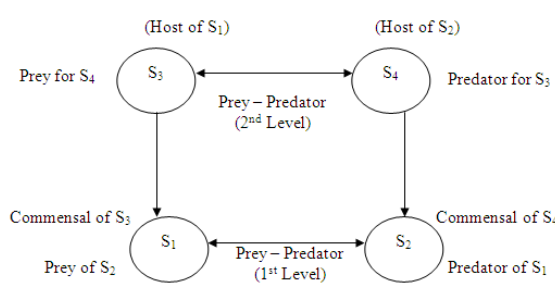

Fig 1. Schematic Sketch of the Syn Eco – System

BASIC EQUATIONS

The model equations for a four species syn eco-system is given by the following system of first order non-linear ordinary differential equations employing the following notation.

Notation

S_1 : Prey for S_2 and commensal for S_3.
S_2 : Predator surviving upon S_1 and commensal for S_4.
S_3 : Host for the commensal (S_1) and Prey for S_4.
S_4 : Host of the commensal (S_2) and Predator surviving upon S_4.
$N_i(t)$: The Population strength of S_i at time t, i = 1, 2, 3, 4
t : Time instant
a_i : Natural growth rate of S_i, i = 1, 2, 3, 4
a_{ii} : Self inhibition coefficient of S_i, i = 1, 2, 3, 4
a_{12},a_{21} :Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{34},a_{43} :Interaction (Prey-Predator) coefficients of S_3 due to S_4 and S_4 due to S_3
a_{13}, a_{24} :Coefficients for commensal for S_1 due to the Host S_3 and S_2 due to the Host S_4
$K_i = \frac{a_i}{a_{ii}}$:Carrying capacities of S_i, i = 1, 2, 3, 4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}, a_{34}, a_{43} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 + a_{24}N_2N_4 \tag{2.2}$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 - a_{34}N_3N_4 \tag{2.3}$$

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 + a_{43}N_3N_4 \tag{2.4}$$

EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1,2,3,4 \tag{3.1}$$

as given in the following Table.

S.No.	Equilibrium State	Equilibrium Point
1	Fully Washed out state	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
2	Only the Host (S_4)of S_2 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
3	Only the Host (S_3)of S_1 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
4*	Only the Predator (S_2) survives	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$
5*	Only the Prey (S_1) survives	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
6	Prey (S_1) and Predator (S_2) washed out	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{\alpha}{\beta}, \overline{N_4} = \frac{\gamma}{\beta}$ where $\alpha = a_3a_{44} - a_4a_{34},\ \beta = a_{33}a_{44} + a_{34}a_{43} > 0$ $\gamma = a_3a_{43} + a_4a_{33} > 0$
7	Prey (S_1) and Host (S_3) of S_1 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{\delta_1}{a_{22}a_{44}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$ where $\delta_1 = a_2a_{44} + a_4a_{24} > 0$
8	Prey (S_1) and Host (S_4) of S_2 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
9	Predator (S_2) and Host (S_3) of S_1 washed out	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
10	Predator (S_2) and Host (S_4) of S_2 washed out	$\overline{N_1} = \frac{\delta_2}{a_{11}a_{33}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$ where $\delta_2 = a_1a_{33} + a_3a_{13} > 0$
11*	Prey (S_1) and Predator (S_2)survives	$\overline{N_1} = \frac{\alpha_1}{\beta_1}, \overline{N_2} = \frac{\gamma_1}{\beta_1}, \overline{N_3} = 0, \overline{N_4} = 0$ where $\alpha_1 = a_1a_{22} - a_2a_{12},\ \beta_1 = a_{11}a_{22} + a_{12}a_{21} > 0$

		$\gamma_1 = a_1 a_{21} + a_2 a_{11} > 0$
12	Only the Prey (S_1) washed out	$\overline{N_1} = 0, \overline{N_2} = \dfrac{a_2\beta + a_{24}\gamma}{a_{22}\beta}, \overline{N_3} = \dfrac{\alpha}{\beta}, \overline{N_4} = \dfrac{\gamma}{\beta}$
13	Only the predator (S_2) washed out	$\overline{N_1} = \dfrac{a_1\beta + a_{13}\alpha}{a_{11}\beta}, \overline{N_2} = 0, \overline{N_3} = \dfrac{\alpha}{\beta}, \overline{N_4} = \dfrac{\gamma}{\beta}$
14	Only the Host (S_3) of S_1 washed out	$\overline{N_1} = \dfrac{a_1 a_{22} a_{44} - a_{12}\delta_1}{a_{44}\beta_1}, \overline{N_2} = \dfrac{a_1 a_{21} a_{44} + a_{11}\delta_1}{a_{44}\beta_1},$ $\overline{N_3} = 0, \overline{N_4} = \dfrac{a_4}{a_{44}}$
15	Only the Host (S_4) of S_2 washed out	$\overline{N_1} = \dfrac{a_{22}\delta_2 - a_2 a_{12} a_{33}}{a_{33}\beta_1}, \overline{N_2} = \dfrac{a_{21}\delta_2 + a_2 a_{11} a_{33}}{a_{33}\beta_1},$ $\overline{N_3} = \dfrac{a_3}{a_{33}}, \overline{N_4} = 0$
16	The co-existent state (or) Normal steady state	$\overline{N_1} = \dfrac{a_{22}\alpha_2 - a_{12}\gamma_2}{\beta_1}, \overline{N_2} = \dfrac{a_{11}\gamma_2 + a_{21}\alpha_2}{\beta_1},$ $\overline{N_3} = \dfrac{\alpha}{\beta}, \overline{N_4} = \dfrac{\gamma}{\beta}$ *where* $\alpha_2 = a_1 + a_{13}\dfrac{\alpha}{\beta},\ \gamma_2 = a_2 + a_{24}\dfrac{\gamma}{\beta} > 0$

The present paper deals with the 2nd level Prey-Predator washed out states only (Sr. Nos. 4, 5, 11 marked * in the above Table -1). The stability of the other equilibrium states will be presented in the forth coming communications.

STABILITY OF THE EQUILIBRIUM STATES

Let $N = (N_1, N_2, N_3, N_4) = \bar{N} + U$ (4.1)

where $U = (u_1, u_2, u_3, u_4)$ is a perturbation over the equilibrium state $\bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4)$.

The basic equations (2.1), (2.2), (2.3), (2.4) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & -a_{12}\bar{N}_1 & a_{13}\bar{N}_1 & 0 \\ a_{21}\bar{N}_1 & a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 + a_{24}\bar{N}_4 & 0 & a_{24}\bar{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\bar{N}_3 - a_{34}\bar{N}_3 & -a_{34}\bar{N}_3 \\ 0 & 0 & a_{34}\bar{N}_4 & a_4 - 2a_{44}\bar{N}_4 + a_{43}\bar{N}_3 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is

$$\det[A - \lambda I] = 0 \tag{4.4}$$

The equilibrium state is stable, if both the roots of the equation (4.4) are negative in case they are real or have negative real parts in case they are complex.

STABILITY OF THE HOST (S_3) OF S_1 AND HOST (S_4) of S_2 WASHED OUT EQUILIBRIUM STATES: (Sl. Nos. 4,5,11 marked * in Table .1)

Equilibrium point

$\bar{N}_1 = 0, \bar{N}_2 = \dfrac{a_2}{a_{22}}, \bar{N}_3 = 0, \bar{N}_4 = 0$:

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = b_1 u_1,\ \frac{du_2}{dt} = k_2 a_{21} u_1 - a_2 u_2 + k_2 a_{24} u_4 \tag{5.1.1}$$

$$\frac{du_3}{dt} = a_3 u_3,\ \frac{du_4}{dt} = a_4 u_4 \tag{5.1.2}$$

Here $b_1 = a_1 - k_2 a_{12}$ (5.1.3)

The characteristic equation for which is

$$(\lambda - b_1)(\lambda + a_2)(\lambda - a_3)(\lambda - a_4) = 0 \tag{5.1.4}$$

Two of the four roots a_3, a_4 are positive and $-a_2$ is negative. Hence the state is unstable.

Case (A): If $b_1 > 0$ (i.e. $a_1 > k_2 a_{12}$)

The solutions of the equations (5.1.1), (5.1.2) are

$$u_1 = u_{10} e^{b_1 t},\ u_2 = Ce^{b_1 t} + (u_{20} - C - D) e^{-a_2 t} + De^{a_4 t} \tag{5.1.5}$$

$$u_3 = u_{30} e^{b_1 t},\ u_4 = u_{40} e^{a_4 t} \tag{5.1.6}$$

Hence $C = \dfrac{k_2 a_{24} u_{10}}{b_1 + a_2}, D = \dfrac{k_2 a_{24} u_{40}}{a_2 + a_4}$ (5.1.7) and $u_{10}, u_{20}, u_{30}, u_{40}$ are the initial values of u_1, u_2, u_3, u_4 respectively.

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations. And the solution curves are illustrated in Figures (2) to (4) and the conclusions are presented here.

Case (i): If $u_{10} < u_{40} < u_{20} < u_{30}$ and $a_4 < b_1 < a_2 < a_3$

In this case the natural birth rates of the Host (S_4) of S_2, Prey (S_1), Predator (S_2) and the Host (S_3) of S_1 are in ascending order. Initially the Host (S_4) of S_2 dominates over the Prey (S_1) till the time instant t_{14}^* and thereafter the dominance is reversed. The time t_{14}^* may be called the dominance time of S_4 over S_1

Here $t_{14}^* = \dfrac{1}{b_1 - a_4} \log\left(\dfrac{u_{40}}{u_{10}}\right)$ (5.1.8)

Case (ii): If $u_{20} < u_{40} < u_{30} < u_{10}$ and $b_1 < a_3 < a_2 < a_4$

In this case the natural birth rates of the Prey (S_1), Host (S_3) of S_1, Predator (S_2) and the Host (S_4) of S_2 are in ascending order. Initially the Prey (S_1) dominates over the Host (S_4) of S_2, Predator (S_2), Host (S_3) of S_1 till the time instant $t_{41}^*, t_{21}^*, t_{31}^*$ respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Host (S_4) of S_2, Predator (S_2) till the time instant t_{43}^*, t_{23}^* respectively and the dominance gets reversed thereafter.

Here $t_{31}^* = \dfrac{1}{b_1 - a_3} \log\left(\dfrac{u_{30}}{u_{10}}\right), t_{43}^* = \dfrac{1}{a_3 - a_4} \log\left(\dfrac{u_{40}}{u_{30}}\right)$ (5.1.9)

Case (iii): If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_3 < a_2 < b_1 < a_4$

In this case the natural birth rates of the Host (S_3) of S_1, Predator (S_2), Prey (S_1) and the Host (S_4) of S_2 are in ascending order. Initially the Prey (S_1) and Predator (S_2) dominates over the Host (S_4) of S_2 till the time instant t_{42}^* and t_{41}^* respectively and thereafter the dominance is reversed.

Case (B): If $b_1 < 0$ (i.e., $a_1 < k_2 a_{12}$)

The solutions in this case are same as in case (A) and the solution curves are illustrated in Figures (5) to (7) and the conclusions are presented here.

Case (i): If $u_{10} < u_{30} < u_{20} < u_{40}$ and $a_4 < b_1 < a_3 < a_2$

In this case the natural birth rates of the Prey (S_1), Host (S_4) of S_2, Host (S_3) of S_1 and the Predator (S_2) are in ascending order. Initially the Host (S_4) of S_2 dominates over the Predator (S_2), Host (S_3) of S_1 till the time instant t_{24}^*, t_{34}^* respectively and thereafter the dominance is reversed.

Case (ii): If $u_{30} < u_{10} < u_{40} < u_{20}$ and $a_3 < a_2 < a_4 < b_1$

In this case the natural birth rates of the Prey (S_1), Host (S_3) of S_1, Predator (S_2) and the Host (S_4) of S_2 are in ascending order. Initially the Prey (S_1) dominates its Host (S_3) till the time instant t_{31}^* and thereafter the dominance is reversed. Also the Predator (S_2) dominates its Host (S_4) till the time instant t_{42}^* and the dominance gets reversed thereafter.

Case (iii): If $u_{40} < u_{20} < u_{30} < u_{10}$ and $b_1 < a_4 < a_2 < a_3$

In this case the natural birth rates of the Prey (S_1), Host (S_4) of S_2, Predator (S_2) and the Host (S_3) of S_1 are in ascending order. Initially the Prey (S_1) dominates over the Host (S_3) of S_1, Predator (S_2) and Host (S_4) of S_2 till the time instant t_{31}^*, t_{21}^* and t_{41}^* respectively and thereafter the dominance is reversed.

Trajectoreis of perturbations

The trajectories in the $u_1 - u_3$ plane given by $x^{a_3} = y_1^{b_1}$ and are shown in Fig.8 and the trajectories in the other planes are

$$x^{a_4} = y_3^{b_1},\ y_2^{a_4} = y_3^{a_3},\ y_1 = C_1 x^{\frac{-a_2}{b_1}} + D_1 x + E_1 x^{\frac{a_4}{b_1}} \quad (5.1.11)$$

$$y_1 = C_1 y_2^{\frac{-a_2}{a_3}} + D_1 y_2^{\frac{b_1}{a_3}} + E_1 y_2^{\frac{a_4}{a_3}},\ y_1 = C_1 y_3^{\frac{-a}{a_4}} + D_1 y_3^{\frac{b_1}{a_4}} + E_1 y_3 \quad (5.1.12)$$

where $C_1 = \dfrac{u_{20} - C - D}{u_{20}}, D_1 = \dfrac{C}{u_{20}}, E_1 = \dfrac{D}{u_{20}}$ (5.1.13)

and $x = \dfrac{u_1}{u_{10}},\ y_1 = \dfrac{u_2}{u_{20}},\ y_2 = \dfrac{u_3}{u_{30}},\ y_3 = \dfrac{u_4}{u_{40}}$ (5.1.14)

Equilibrium point

$$\overline{N}_1 = \frac{a_1}{a_{11}}, \overline{N}_2 = 0, \overline{N}_3 = 0, \overline{N}_4 = 0.$$

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = -a_1 u_1 - k_1 a_{12} u_2 + k_1 a_{13} u_3,\ \frac{du_2}{dt} = b_2 u_2 \quad (5.2.1)$$

$$\frac{du_3}{dt} = a_3 u_3,\ \frac{du_4}{dt} = a_4 u_4 \quad (5.2.2)$$

Here $b_2 = a_2 + k_1 a_{21} > 0$ (5.2.3)

The characteristic equation for which is

$$(\lambda + a_1)(\lambda - b_2)(\lambda - a_3)(\lambda - a_4) = 0 \quad (5.2.4)$$

Three of the four roots b_2, a_3, a_4 are positive and $-a_1$ is negative. Hence the state is unstable and the solutions of the equations (5.2.1), (5.2.2) are

$$u_1 = (u_{10} + R - S)\, e^{-a_1 t} - \mathrm{Re}^{b_2 t} + Se^{a_3 t}, \; u_2 = u_{20}\, e^{b_2 t} \quad (5.2.5)$$

$$u_3 = u_{30}\, e^{a_3 t}, \; u_4 = u_{40}\, e^{a_4 t} \quad (5.2.6)$$

Here $R = \dfrac{k_1 a_{12} u_{20}}{a_1 + b_2} > 0, \; S = \dfrac{k_1 a_{13} u_{30}}{a_1 + a_3} > 0$ (5.2.7)

The solution curves are illustrated in Figures (9), (10) and the conclusions are presented here.

Case (i): If $u_{10} < u_{30} < u_{20} < u_{40}$ and $b_2 < a_3 < a_1 < a_4$

In this case the natural birth rates of the Predator (S_2), Host (S_3) of S_1, Prey (S_1) and the Host (S_4) of S_2 are in ascending order. Initially the Predator (S_2) dominates over the Prey (S_1), Host (S_3) of S_1 till the time instant t^*_{12}, t^*_{32} respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Prey (S_1) till the time instant t^*_{13} and the dominance gets reversed thereafter.

Here $t^*_{32} = \dfrac{1}{b_2 - a_3} \log\left(\dfrac{u_{30}}{u_{20}}\right)$ (5.2.8)

Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_4 < b_2 < a_1 < a_3$

In this case the natural birth rates of the Host (S_4) of S_2, Predator (S_2), Prey (S_1) and the Host (S_3) of S_1 are in ascending order. Initially the Host (S_4) of S_2 dominates over the Host (S_3) of S_1, Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates its Host (S_3) till the time instant t^*_{31} and the dominance gets reversed thereafter.

Here $t^*_{34} = \dfrac{1}{a_3 - a_4} \log\left(\dfrac{u_{40}}{u_{30}}\right), t^*_{24} = \dfrac{1}{b_2 - a_4} \log\left(\dfrac{u_{40}}{u_{20}}\right)$ (5.2.9)

Trajectories of Perturbations

The trajectories in the $u_2 - u_3$ plane given by

$$y_1^{a_3} = y_2^{b_2} \quad (5.2.10)$$

and are shown in Fig.11 and the trajectories in the other planes are

$$y_1^{a_4} = y_3^{b_2}, \; y_2^{a_4} = y_3^{a_3}, \; x = R_1 y_1^{\frac{-a_1}{b_2}} - R_2 y_1 + R_3 y_1^{\frac{a_3}{b_2}} \quad (5.2.11)$$

$$x = R_1 y_2^{\frac{-a_1}{a_3}} - R_2 y_2^{\frac{b_2}{a_3}} + R_3 y_2, \; x = R_1 y_3^{\frac{-a_1}{a_4}} - R_2\, y_3^{\frac{b_2}{a_4}} + R_3 y_3^{\frac{a_3}{a_4}} \quad (5.2.12)$$

where $R_1 = \dfrac{u_{10} + R - S}{u_{10}}, R_2 = \dfrac{R}{u_{10}}, R_3 = \dfrac{S}{u_{10}}$ (5.2.13)

Equilibrium point

$$\overline{N}_1 = \frac{\alpha_1}{\beta_1}, \overline{N}_2 = \frac{\gamma_1}{\beta_1}, \overline{N}_3 = 0, \overline{N}_4 = 0:$$

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = d_1 u_1 - a_{12} \frac{\alpha_1}{\beta_1} u_2 + a_{13} \frac{\alpha_2}{\beta_1} u_3 \quad (5.3.1)$$

$$\frac{du_2}{dt} = a_{21} \frac{\gamma_1}{\beta_1} u_1 + d_2 u_2 + a_{24} \frac{\gamma_1}{\beta_1} u_4 \quad (5.3.2)$$

$$\frac{du_3}{dt} = a_3 u_3, \frac{du_4}{dt} = a_4 u_4 \quad (5.3.3)$$

Here $d_1 = a_1 - 2a_{11} \dfrac{\alpha_1}{\beta_1} - a_{12} \dfrac{\gamma_1}{\beta_1}, \; d_2 = a_2 - 2a_{22} \dfrac{\gamma_1}{\beta_1} + a_{21} \dfrac{\alpha_1}{\beta_1}$ (5.3.4)

The characteristic equation for which is

$$\left[\lambda^2 - (d_1 + d_2)\,\lambda + \left(d_1 d_2 + a_{12} a_{21} \frac{\alpha_1 \gamma_1}{\beta_1^2}\right)\right](\lambda - a_3)(\lambda - a_4) = 0 \quad (5.3.5)$$

Two of the four roots a_3, a_4 are positive. Hence the state is unstable. Let λ_1, λ_2 be the zeros of the quadratic polynomial on the LH.S of the equation (5.3.5)

Case (A):

When $\alpha_1 < 0$ $\left\{\text{ie,} \left[(d_1 - d_2)^2 - 4a_{12}\, a_{21} \dfrac{\alpha_1 \gamma_1}{\beta_1^2}\right] > 0\right\}$

and $\alpha_1 > 0, (d_1 - d_2)^2 > 4a_{12}\, a_{21} \dfrac{\alpha_1 \gamma_1}{\beta_1^2}$

One root λ_1 is positive while the other root λ_2 negative. The solutions of the equations (5.3.1), (5.3.2), (5.3.3) are

$$u_1 = \left[\frac{\left(M_1 - N_1 - a_{12} \frac{\alpha_1}{\beta_1} u_{20}\right) - (M - N - u_{10})(d_1 - \lambda_2)}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t}$$

$$+ \left[\frac{\left(M_1 - N_1 - a_{12} \frac{\alpha_1}{\beta_1} u_{20}\right) - (M - N - u_{10})(d_1 - \lambda_1)}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} + Me^{a_3 t} - Ne^{a_4 t} \quad (5.3.6)$$

$$u_2 = \frac{\beta_1}{a_{12}\alpha_1} \left\{\left[\left[\frac{\left(M_1 - N_1 - a_{12} \frac{\alpha_1}{\beta_1} u_{20}\right) - (M - N - u_{10})(d_1 - \lambda_2)}{\lambda_1 - \lambda_2} (d_1 - \lambda_1)\right] e^{\lambda_1 t}\right.\right.$$

$$+\left[\left[\frac{\left(M_1-N_1-a_{12}\frac{\alpha_1}{\beta_1}u_{20}\right)-(M-N-u_{10})(d_1-\lambda_1)}{\lambda_2-\lambda_1}(d_1-\lambda_2)\right]e^{\lambda_2 t}+M_1e^{a_3t}+N_1e^{a_4t}\right\} \quad (5.3.7)$$

$$u_3 = u_{30}e^{a_3 t},\ u_4 = u_{40}e^{a_4 t} \quad (5.3.8)$$

Here

$$\bar{\alpha}=a_{13}\frac{\alpha_1}{\beta_1}(a_3-d_2)u_{30},\ \bar{\beta}=a_{12}a_{24}\frac{\alpha_1\gamma_1}{\beta_1^2}u_{40},\ \bar{\gamma}=d_1d_2+a_{12}a_{21}\frac{\alpha_1\gamma_1}{\beta_1^2} \quad (5.2.9)$$

$$M=\frac{\bar{\alpha}}{a_3^2-(d_1+d_2)a_3+\bar{\gamma}},\ N=\frac{\bar{\beta}}{a_4^2-(d_1+d_2)a_4+\bar{\gamma}} \quad (5.3.10)$$

$$M_1 = M(d_1-a_3)+a_{13}\frac{\alpha_1}{\beta_1}u_{30},\ N_1 = N(a_4-d_1) \quad (5.3.11)$$

The solution curves are illustrated in Figures 12, 14 and the conclusions are presented here.

Case (i): If $u_{10} < u_{40} < u_{30} < u_{20}$ and $a_1 < a_4 < a_3 < a_2$

In this case the Prey (S_1) has the least natural birth rate and the Predator (S_2), dominates the Host (S_3) of S_1, Host (S_4) of S_2, Prey (S_1) in natural growth rate as well as in its population strength.

Case (ii): If $u_{20} < u_{10} < u_{40} < u_{30}$ and $a_1 < a_4 < a_2 < a_3$

In this case the natural birth rates of the Prey (S_1), Host (S_4) of S_2, Predator (S_2) and the Host (S_3) of S_1 are in ascending order. Initially the Host (S_4) of S_2, Prey (S_1) dominates over the Predator (S_2) till the time instant t_{34}^*, t_{31}^* respectively and thereafter the dominance is reversed.

Case (iii): If $u_{30} < u_{20} < u_{40} < u_{10}$ and $a_4 < a_1 < a_3 < a_2$

In this case the natural birth rates of the Host (S_4) of S_2, Prey (S_1), Host (S_3) of S_1 and the Predator (S_2) are in ascending order. Initially the Prey (S_1) dominates over the Predator (S_2), Host (S_3) of S_1 till the time instant t_{21}^*, t_{31}^* respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Predator (S_2), Host (S_3) of S_1 till the time instant t_{24}^*, t_{34}^* respectively and the dominance gets reversed thereafter.

Case (B): When $\alpha_1 > 0$ and $(d_1-d_2)^2 < 4a_{12}a_{21}\frac{\alpha_1\gamma_1}{\beta_1^2}$

The roots λ_1, λ_2 are complex and the solutions in this case are same as in case (A). This is illustrated in Fig. 15

Trajectories of Perturbations

The trajectories in the $u_3 - u_4$ plane given by

$$y_2^{a_4} = y_3^{a_3} \quad (5.3.12)$$

and are shown in Fig.16 and the trajectories in the other planes are

$$x = A_1 y_2^{\frac{\lambda_1}{a_3}} + A_2 y_2^{\frac{\lambda_2}{a_3}} + \frac{M}{u_{10}} y_2 - \frac{N}{u_{10}} y_2^{\frac{a_4}{a_3}},\ x = A_1 y_3^{\frac{\lambda_1}{a_4}} + A_2 y_3^{\frac{\lambda_2}{a_4}} + \frac{M}{u_{10}} y_3^{\frac{a_3}{a_4}} - \frac{N}{u_{10}} y_3 \quad (5.3.13)$$

$$y_1 = A_3 y_2^{\frac{\lambda_1}{a_3}} + A_4 y_2^{\frac{\lambda_2}{a_3}} + \bar{M}_1 y_2 + \bar{N}_1 y_2^{\frac{a_4}{a_3}},\ y_1 = A_3 y_3^{\frac{\lambda_1}{a_4}} + A_4 y_3^{\frac{\lambda_2}{a_4}} + \bar{M}_1 y_3^{\frac{a_3}{a_4}} + \bar{N}_1 y_3 \quad (5.3.14)$$

where

$$A_1 = \frac{\left(M_1-N_1-a_{12}\frac{\alpha_1}{\beta_1}u_{20}\right)-(M-N-u_{10})(d_1-\lambda_2)}{u_{10}(\lambda_1-\lambda_2)} \quad (5.3.15)$$

$$A_2 = \frac{\left(M_1-N_1-a_{12}\frac{\alpha_1}{\beta_1}u_{20}\right)-(M-N-u_{10})(d_1-\lambda_1)}{u_{10}(\lambda_2-\lambda_1)} \quad (5.3.16)$$

$$A_3 = \frac{\beta_1(d_1-\lambda_1)\left[\left(M_1-N_1-a_{12}\frac{\alpha_1}{\beta_1}u_{20}\right)-(M-N-u_{10})(d_1-\lambda_1)\right]}{a_{12}\alpha_1 u_{10}(\lambda_1-\lambda_2)} \quad (5.3.17)$$

$$A_4 = \frac{\beta_1(d_1-\lambda_2)\left[\left(M_1-N_1-a_{12}\frac{\alpha_1}{\beta_1}u_{20}\right)-(M-N-u_{10})(d_1-\lambda_1)\right]}{a_{12}\alpha_1 u_{10}(\lambda_2-\lambda_1)} \quad (5.3.18)$$

$$\bar{M}_1 = \frac{\beta_1 M_1}{a_{12}\alpha_1},\ \bar{N}_1 = \frac{\beta_1 N_1}{a_{12}\alpha_1} \quad (5.3.19)$$

Perturbation Graphs

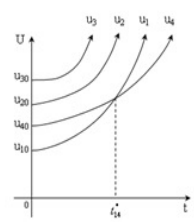
Fig. 2

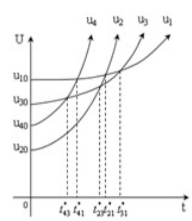
Fig. 3

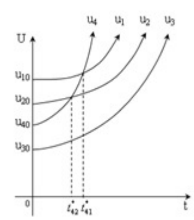
Fig. 4

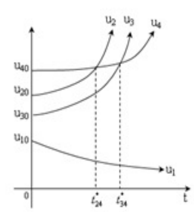
Fig. 5

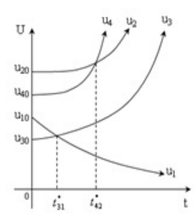
Fig. 6

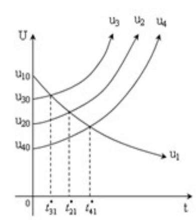
Fig. 7

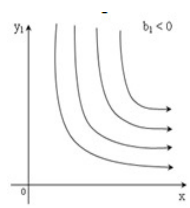
Fig. 8

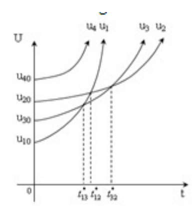
Fig. 9

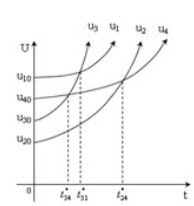
Fig. 10

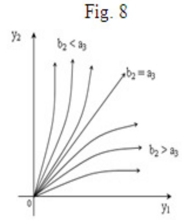
Fig. 11

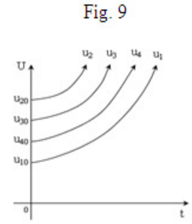
Fig. 12

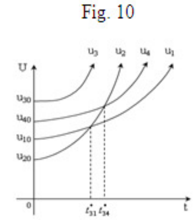
Fig. 13

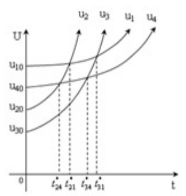

Fig. 14

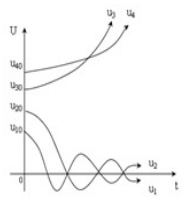

Fig. 15

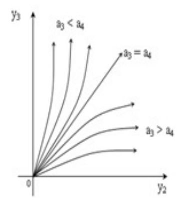

Fig. 16

CONCLUSION

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between four species (S_1, S_2, S_3, S_4) with the population relations.

S_1 a Prey to S_2 and Commensal to S_3, S_2 is a Predator living on S_1 and Commensal to S_4, S_3 a Host to S_1, S_4 a Host to S_2 and S_3 a Prey to S_4, S_4 a Predator to S_3.

The present paper deals with the study on stability of 2nd level Prey-Predator washed out states only of the above problem. It is observed that the 2nd level Prey Predator Washed out States are unstable. The stability of the other equilibrium states were already investigated and communicated to several international journals.

ACKNOWLEDGEMENT

I thank to Professor (Retd), N.Ch.Pattabhi Ramacharyulu, Department of Mathematics, National Institute of Technology, Warangal (A.P.), India for his valuable suggestions and encouragement. And I acknowledge to Mr.K.Ravindranath Gupta for neat typing of this research paper.

REFERENCES

[1] Lotka A. J. 1925. Elements of Physical Biology, Williams & Wilking, Baltimore.

[2] Volterra V.1931. Leconssen La Theorie Mathematique De La Leitte Pou Lavie, Gauthier-Villars, Paris.

[3] May.R.M. 1973. Stability and Complexity in Model Eco-systems, Princeton University Pres, Princeton.

[4] Smith.J.M. 1974. Models in Ecology, Cambridge University Press, Cambridge.

[5] Kushing J.M. 1977. Integro-Differential Equations and Delay Models in Population Dynamics, Lecture Notes in Bio-Mathematics, Springer Verlag, 20.

[6] Kapur J. N. 1985. Mathematical Modelling in Biology and Medicine, Affiliated East West.

[7] Srinivas N. C.1991. "Some Mathematical Aspects of Modeling in Bio-medical Sciences" Ph.D Thesis, Kakatiya University,

[8] Lakshmi Narayan K. & Pattabhiramacharyulu. N. Ch. 2007. "A Prey-Predator Model with Cover for Prey and Alternate Food for the Predator and Time Delay," International Journal of Scientific Computing. 1, 7-14.

[9] Archana Reddy .R. 2010. "On the stability of some Mathematical Models in BioSciences – Interacting Species", Ph.D.Thesis, J.N.T.U.

[10] Bhaskara Rama Sharma .B. 2010. "Some Mathematical Models in Competitive Eco-Systems", Ph.D. Thesis, Dravidian University.

[11] Phani Kumar and Pattabhi Ramacharyulu N.Ch. 2010. "Three Species Eco-System Consisting of a Prey, Predator and a Host Commensal to the Prey", "Int.J.Open Problems Compt.Math"., Vol.3, No.1.

[12] Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch. 2010. "On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-II" (Prey and Predator washed out states) "International eJournal of Mathematics & Engineering", 5, 60 - 74.

[13] Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch. 2011."On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-VII" (Host of the Prey Washed Out States,) "International Journal of Applied Mathematical Analysis and Applications, Vol.6, No.1-2, pp.85 – 94.

[14] Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch. 2011. "On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-VIII" (Host of the Predator Washed Out States) "Advances in Applied Science, Research, 2(5), pp.197-206.

[15] Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch.2010. "On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts-I (Fully Washed Out State)", Global Journal of Mathematical Sciences : Theory and Practical, Vol.2, No.1, pp.65 – 73.

[16] Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch. 2011. "On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts-V (Predator Washed Out States)", Int. J. Open Problems Compt. Math. Vol.4, No.3, pp.129 – 145.

[17] Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch. 2011. "On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts-VII (Host of S_2 Washed Out States)", Journal of Communication and Computer, Vol.8, No.6, pp.415 – 421.