Journal of Experimental Sciences Vol. 1, Issue 5, Pages 25-27 [2010]

JES Formal Sciences

Regular Article

# Mathematical Science Reformulate Convex Relaxation in Logic Based Optimization

**Rajeev Kishore***

Department of Mathematic, Jamuna Prasad Memorial Degree College, Bareilly

## Abstract

We present a convex conic Relaxation for a problem of maximizing an indefinite quadratic form over a set of, convex constraints on the squared variables, convex envelopes of non convex functions are widely used to calculate lower bounds of solutions of non linear programming problem (NLP). This paper proposes a non linear continuous convex envelope for (Linear, bilinear, trilinear fractional) monomial terms of odd degree and the range of variables (x, y, z) includes zero.
We also drive a linear relaxation from the proposed envelope and computer both the Linear and non linear formulations with relaxations obtained using other approaches.

**Keywords**: Odd degree, Monomial, Convex relaxation, Global optimization

## Introduction

Our work has considered the derivation of convex relaxations for monomial terms of odd degree when the variables range includes zero. The main innovation of the $\alpha\beta\beta$ algorithm is in the underestimations of a general non convex function.
The convex relaxation is built in two stages. First the problem is reduced to a standard form where nonlinear terms of the same type are collected in lists, then each non linear term is replaced by the corresponding convex under and overestimations. Convexification is the second stage of the process where the actual convex relaxation of the original problem with in the current region $[x^U, x^U]$ is built. Where $x^L, x^U$ are the variable bounds L and U are the linear constraints bounds.

### Convex relaxations

A relaxation cannot be used to solve a difficult problem directly because the solution of the original problem cannot, in general, be directly inferred from the solution of the solution.
Relaxations are however, very important in the field of deterministic global optimization. One of the most important tools in this field is the Branch and Bound algorithm, which uses a convex (or linear) relaxation at each step to calculate the Lower bound in a region. Convex relaxations for non convex problems are obtained by substituting the (non convex) objective function f(x) with a convex relaxation $\underline{f}(x)$ and the (non-convex) feasible region $\Omega$ with a convex set $\overline{\Omega}$ such that $\Omega \leq \overline{\Omega}$.

### Linear relaxations

It is possible to use Linear over and underestimates for each non linear term in a nonconvex NLP in order to obtain a linear relaxation of the problem because a linear problem is always convex, the convexity properties that guarantee the validity of a lower bound remain true. The advantage of a Linear relaxation with respect to a convex relaxation is that Linear optimization software can be employed to solve the relaxed understanding problem. Linear optimization codes are much more efficient than non linear optimization software, hence the overall run of a branch and Bound algorithm must be faster.

### $\alpha\beta\beta$ Convex relaxation

Floudas and Co- workers have proposed a Branch and Bound algorithm (Called $\alpha\beta\beta$ (1, 2, 3, 4) ) for general non-convex twice- differentiable problems. The algorithm aims to solve a Problem in the given form.

$$\min_{x \in \Omega} f(x)$$

Where $\Omega$ is the feasible region and $\Omega$ is defined as

$$\Omega = \{x \in R^n \mid \forall_i \leq m(n_i(x)=0) \wedge \forall_i \leq [m' g_i(x) \leq 0] \wedge x^2 \leq x \leq x^\mu\}$$

Any non linear, twice-differential function f(x) in the problem, be the object function or One of the constraints, can be reformulated exactly as

$$f(x) = C^T x + f_c(x)$$

$$+\sum_i b_i x_{B_{1(i)}} x_{B_{2(i)}} + \sum_i t_i x_{T_{1(i)}} x_{T_{2(i)}} x_{T_{3(i)}} + \sum_i d_i \frac{x_{f_{1(i)}}}{x_{f_{2(i)}}}$$

$$+\sum_i d_i \frac{x_{R_{1(i)}} + x_{R_{2(i)}}}{x_{R_{3(i)}}} + \sum_i f_{U_{(i)}}(x_i) + \sum_i f_{N_{(i)}}(x)$$

Where

$C, x \in R^n$ and each $b_i, t_i, d_i, r_i$ is a real constant;

$f_c(x)$ is a general Convex function

$B_j, T_j, f_j, R_j, U_j, N_j$ are integer functions $N \rightarrow [1, .........., n]$

Each $f_{U(t)}$ is a concave univariate function term;

Each $f_{N(t)}$ is a general non-convex function term.

### Relaxation for bilinear term xy

For a bilinear term xy, McCormick's under estimators [5] are used. A New variable $W_B$ is added to the problem (it replaces the bilinear term xy) and following inequality constrains are instead in the relaxed problem

$$W_B \geq x^L y + y^L x - x^L y^L$$

$$W_B \geq x^U y + y^U x - x^U y^U$$

$$W_B \leq x^U y + y^L x - x^U y^L$$

$$W_B \leq x^L y + y^U x - x^L y^U$$

* Corresponding Author, Email: Rajeevkishorepandey@rediffmail.com



ISSN: 2218-1768

The above Linear inequalities have been shown to be the convex envelope of a linear term [6]. The maximum separation of the linear term xy from its convex envelope max $(x^L y + y^L x - x^L y^L, x^U y + y^U x - x^U y^U)$ inside the rectangle

$[x^L, x^U] \times [y^L, y^U]$ Occurs at the middle point $[\frac{X^L + X^U}{2}, \frac{Y^L + Y^U}{2}]$ and is equal to $\frac{(x^U - x^L)(y^U - y^L)}{4}$ [7]

**Relaxation for trilinear term in xyz**

For a trilinear $xyz$ a new variable $W_T$ is introduced to replace the trilinear term xyz, together with the following constraints [8].

$$W_T \geq xy^L z^L + x^L yz^L + x^L y^L z - 2x^L y^L z^L$$
$$W_T \geq xy^U z^U + x^U yz^L + x^U y^L z - x^U y^L z^L - x^U y^U z^U$$
$$W_T \geq xy^L z^L + x^L yz^U + x^L y^U z - x^L y^U z^U - x^L y^L z^L$$
$$W_T \geq xy^U z^L + x^U yz^U + x^L y^U z - x^L y^U z^L - x^U y^U z^U$$
$$W_T \geq xy^L z^U + x^L yz^L + x^U y^L z - x^U y^L z^U - x^L y^L z^L$$
$$W_T \geq xy^L z^U + x^L yz^U + x^U y^U z - x^L y^L z^U - x^U y^U z^U$$
$$W_T \geq xy^U z^L + x^U yz^L + x^L y^L z - x^U y^U z^L - x^L y^L z^L$$
$$W_T \geq xy^U z^U + x^U yz^U + x^U y^U z - 2x^U y^U z^U$$

**Relaxation for fractional term in x/y**

Fractional terms x/y are underestimated by replaced them with a new variable $W_F$ and adding two new constraints to the problem.

$$W_F \geq \begin{cases} \frac{x^L}{y} + \frac{x}{y^U} - \frac{x^L}{y^U} \; if x^L \geq 0 \\ y^U - \frac{x^L y}{y^L y^U} + \frac{x^L}{y^L} \; if x^L < 0 \end{cases}$$

$$W_F \geq \begin{cases} \frac{x^U}{y} + \frac{x}{y^U} - \frac{x^U}{y^L} \; if x^U \geq 0 \\ y^L - \frac{x^U y}{y^L y^U} + \frac{x^U}{y^U} \; if x^U < 0 \end{cases}$$

**Relaxation for fractional Linear terms xy/z**

Fractional Linear terms $xy/z$ can be underestimated by Replacing them by a new variable $W_{FT}$ and adding a new constraints for $(X^L, Y^L, Z^L \geq 0)$

$$W_{FT} \geq \frac{xy^L}{z^U} + \frac{x^L y}{z^U} + \frac{x^L y^L}{z} - 2\frac{x^L y^L}{z^U}$$

$$W_{FT} \geq \frac{xy^L}{z^U} + \frac{x^L y}{z^L} + \frac{x^L y^U}{z} - \frac{x^L y^U}{z^L} - \frac{x^L y^L}{z^U}$$

$$W_{FT} \geq \frac{xy^U}{z^L} + \frac{x^U y}{z^U} + \frac{x^U y^L}{z} - \frac{x^U y^L}{z^U} - \frac{x^U y^U}{z^L}$$

$$W_{FT} \geq \frac{xy^U}{z^U} + \frac{x^U y}{z^L} + \frac{x^L y^U}{z} - \frac{x^L y^U}{z^U} - \frac{x^U y^U}{z^L}$$

$$W_{FT} \geq \frac{xy^L}{z^U} + \frac{x^L y}{z^L} + \frac{x^U y^L}{z} - \frac{x^U y^L}{z^L} - \frac{x^L y^L}{z^U}$$

$$W_{FT} \geq \frac{xy^U}{z^U} + \frac{x^U y}{z^L} + \frac{x^L y^U}{z} - \frac{x^L y^U}{z^U} - \frac{x^U y^U}{z^L}$$

$$W_{FT} \geq \frac{xy^L}{z^U} + \frac{x^L y}{z^L} + \frac{x^U y^L}{z} - \frac{x^U y^L}{z^L} - \frac{x^L y^L}{z^U}$$

$$W_{FT} \geq \frac{xy^U}{z^L} + \frac{x^U y}{z^L} + \frac{x^U y^U}{z} - 2\frac{x^U y^U}{z^L}$$

To relax a concave univariate function $f_{U(i)}(x)$ over $(x^L, x^U)$, the $\alpha\beta\beta$ algorithm uses a chord under estimator

$$f(x^L) + \frac{f(x^U) - f(x^L)}{x^U - x^L}(x - x^L)$$

The main innovation of the $\alpha\beta\beta$ algorithm is in the underestimation of a general non-convex function term $f_{N_{(i)}}(x)$.

This is underestimated over the entire domain $[x^L, x^U] \subseteq R^n$ by the function $L(x)$ defined as follows:

$$L(x) = f(x) + \sum_{i=1}^{n} a_i (x_i^L - x_i)(x_i^U - x_i)$$

Where $a_i$ are positive scalars that are sufficiently large to render the underestimating function convex. A good feature of this kind of under estimator is that, unlike other under estimators, it does not introduces any new variable or constraint, so that the size of the relaxed problem is the same as the size of the original problem regardless of how many non-convex terms it involves.

Since the sum $\sum_{i=1}^{n} a_i (x_i^L - x_i)(x_i^U - x_i)$ is always negative, $L(x)$ is an under estimator for $f(x)$. Since the quadratic term is convex.

## Conclusion

This paper has provided a literature review of various techniques for reformulation of optimization problem. It is clear from the review presented in this paper that much progress has been achieved in both exact reformulation and convex relaxation of non-convex NLPs. Then quadratic, bilinear, trilinear and fractional terms reformulate by convex relaxation in linear form. Convex (Linear) relaxation at each step to calculate the lower bounds region. Thus many engineering optimization problem can be formulated as non-convex non-linear programming problems involving a non-linear objective function subject to non-linear constraints.

## References


1. C. Adjiman Global optimization Techniques for Process Systems Engineering Ph. D. thesis Princeton University, June 1998.
2. C. Adjiman, S. Dallwigo, C. Floudas, and A. Neumaier. A global optimization method, $\alpha\beta\beta$, for general twice differentiable constrained NLPs: I Theortical advances. Computers & chemical Engineering, 22:1137-1158, 1998.
3. C. Adjiman, I. Androulakis, and C. Floudas. A global optimization Method, $\alpha\beta\beta$, for general twice differentiable constrained NLPs: II Implementation and computational

results. Computers & chemical engineering, 22:1159-1179, 1998.

4. C. Adjiman, C. Schweiger and C. fioudas. Mixed-Integer nonlinear optimization in process synthesis. In D. Du and P. (Eds.) Prodalos, editors, handbook of combinatorial ootimimization, vol.I Pages 1-76 dordeeht, 1998. Kluwer Academic Publishers.
5. G. McCormick, Computability of global solutions to factorable nonconvex Programs. Part I- Convex underestimating Problems. Mathematical programming. 10:146-175, 1976.
6. F. Al-khayyal and J. Falk. Jointly constrained biconvex Programming. Mathematics of operation Research, 8:273-286,1983.
7. Androulakis, C. Maranas, and C. floudas $\alpha\beta\beta$ a global optimization method for general constrained non convex problems. Journal of global optimization 7:337-363, 1995.
8. C. Maranas and C. floudas finding all solutions to non linearly constrained systems of equations journal of global optimization.7:143-182,1995.

Please Cite This Article As:
Rajeev Kishore. 2010. Mathematical Science Reformulate Convex Relaxation in Logic Based Optimization. J. Exp. Sci. 1(5):25-27.