

Review Article

Equitable public land partitioning in community agriculture: A Hilbert space framework comparing mechanism design and stochastic-geometric Markov models

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Abstract

Equitable access to public land is not only a technical allocation problem; it also shapes production opportunities, perceived justice, cooperation and the sustainability of community agriculture. This study develops a unified Hilbert-space framework for comparing two approaches to public-land allocation: mechanism-design procedures based on fair division and stochastic-geometric Markov models. The land domain is represented as a probability space and community members' heterogeneous valuations are treated as vector fields. Within this common framework, mechanism design seeks allocations that satisfy fairness constraints while protecting social welfare, whereas the stochastic approach describes allocation as a dynamic process whose long-run behavior is governed by a Markov semigroup. The comparison highlights a practical trade-off. Mechanism design responds more directly and stably to localized valuation bias and is computationally attractive when communities are large and preferences can be elicited reliably. Stochastic-geometric methods are better able to absorb global spatial distortions through ergodic averaging, which may be useful where land quality and spatial conditions are highly heterogeneous. A spectral hybrid operator is therefore proposed to combine the strengths of both approaches. Beyond its mathematical contribution, the framework provides a decision-support perspective for community land governance by linking fairness, robustness and computational feasibility to the characteristics of beneficiaries and agricultural space. It offers a theoretical basis for choosing allocation procedures that can support transparent, equitable and context-sensitive management of public agricultural land.

Keywords: Community agriculture, Equitable land allocation, Fair division, Land governance, Markov models, Rural development, Spectral operator theory

Introduction and problem formalization

Public land allocation in community agriculture is simultaneously a question of production, social welfare and rural governance. Access to land influences what households can produce, the livelihood opportunities available to them, and their perception of whether local institutions distribute productive resources fairly. In practice, beneficiaries rarely value the same parcel in the same way: soil conditions, location, accessibility, water availability and expected productive use can generate heterogeneous preferences. A credible allocation procedure must therefore reconcile individual valuations with collective objectives such as fairness, transparency, efficiency and social stability. The mathematical framework developed below formalizes this socio-economic allocation problem without assuming that community members are homogeneous. Let $(\Omega, \mathcal{F}, \lambda)$ be a standard Borel probability space (Ma, 2021; Cléry & Mazliak, 2022; Harremoës, 2025) with $\Omega \subset \mathbb{R}^2$, $\lambda(\Omega) = 1$, and $\mathcal{F}$ the completion of the Borel σ -algebra. The public land domain admits a finite partition satisfying $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^n A_i = \Omega$

Each community member $i \in N = \{1, \dots, n\}$ is represented by a nonatomic valuation measure $v_i : \mathcal{F} \rightarrow \mathbb{R}_{\geq 0}$ absolutely continuous with respect to λ . In substantive terms, this valuation captures the relative importance that a beneficiary assigns to different parts of the available land. By the Radon-Nikodym theorem (Bose, 2025; Kharazishvili, 2025; Roselli & Willem, 2026) there exists $f_i \in L^2(\Omega, \lambda)$ such that with $dv_i = f_i d\lambda$

$$\|f_i\|_{L^2} \leq V_{\max} < \infty \dots (1)$$

The valuation vector field is $f = (f_1, \dots, f_n) \in H$, where

$$H = L^2(\Omega, \lambda; \mathbb{R}^n) \dots (2)$$

is equipped with the inner product

$$\langle f, g \rangle_H = \sum_{i=1}^n \int_{\Omega} f_i(\omega) g_i(\omega) d\lambda(\omega) \dots (3)$$

and the induced squared norm

$$\|f\|_H^2 = \sum_{i=1}^n \|f_i\|_{L^2}^2 \dots (4)$$

A partition $A = (A_1, \dots, A_n)$ induces the characteristic vector field $\chi_A(\omega) = (\chi_{A_1}(\omega), \dots, \chi_{A_n}(\omega))$ satisfying $\chi_A(\omega) \in \{0, 1\}^n$ and

$$\sum_{i=1}^n \chi_{A_i}(\omega) = 1 \text{ } \lambda\text{-a.e.} \dots (5)$$

The utility of agent i for partition A is

$$u_i(A) = \int_{A_i} f_i d\lambda = \langle f_i, \chi_{A_i} \rangle_{L^2} \dots (6)$$

The space of admissible partitions is

$$\Delta_n(F) = \{A \in F^n: A_i \cap A_j = \emptyset, \bigcup_{i=1}^n A_i = \Omega\} \dots (7)$$

which embeds into H via $A \mapsto \chi_A$.

The Nash social welfare (NSW) (Jain & Vaish, 2023; Wu *et al.*, 2020) and utilitarian welfare (UW) (Echenique & Valenzuela-Stookey, 2026) for a partition A are

$$NSW(A) = \prod_{i=1}^n u_i(A) \dots (8)$$

and

$$UW(A) = \sum_{i=1}^n u_i(A) \dots (9)$$

Fairness properties are defined as follows: a partition is envy-free (EF) if

$$u_i(A_i) \geq u_i(A_j) \text{ for all } i, j \in N \dots (10)$$

proportional (PROP) if

$$u_i(A_i) \geq \frac{1}{n} v_i(\Omega) \text{ for all } i \in N \dots (11)$$

and equitable (EQ) if

$$u_i(A_i) = u_j(A_j) \text{ for all } i, j \in N \dots (12)$$

The fairness-constraint manifold is

$$\mathcal{M}_{\text{EF}} = \{\chi_A : A \in \Delta_n(\mathcal{F}), u_i(A_i) \geq u_i(A_j) \dots (13)$$

which is closed but nonlinear. Its tangent cone at a point χ_A is

$$T_{\chi_A} \mathcal{M}_{\text{EF}} = \{h \in H : \sum_{i=1}^n h_i = 0 \dots (14)$$

Mechanism design seeks a social choice function $F: H \rightarrow \Delta_n(F)$ maximizing a social objective

$$\Phi(f, A) = \sum_{i=1}^n \varphi_i(\langle f_i, \chi_{A_i} \rangle) \dots (15)$$

subject to F being dominant-strategy incentive-compatible (DSIC). DSIC requires, for every agent i , that

$$\langle f_i, \chi_{F_i(f)} \rangle \geq \langle f_i, \chi_{F_i(f', f_{-i})} \rangle \text{ for all } f'_i \dots (16)$$

where $f_{-i} = (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$. This condition is equivalent to the monotonicity condition for the operator $f \mapsto \chi_{F(f)}$:

$$\langle F(f) - F(f'), f - f' \rangle_H \geq 0 \text{ for all } f, f' \in H \dots (17)$$

The stochastic-geometric approach defines a Markov chain $\{X_t\}_{t \geq 0}$ on the state space $S = \Delta_n(\mathcal{F})$ with transition kernel

$$K(A, dB) = \frac{1}{Z(A)} \exp(-\beta E(B) - \gamma d_H^2(A, B)) \mu_0(dB) \dots (18)$$

where the energy functional E is

$$E(A) = \sum_{i=1}^n \left(\int_{A_i} f_i d\lambda - \bar{u} \right)^2 + \lambda_{\text{reg}} \sum_{i=1}^n \text{Per}(A_i) \dots (19)$$

with

$$\bar{u} = \frac{1}{n} \sum_{j=1}^n \int_{A_j} f_j d\lambda \dots (20)$$

and the perimeter defined by the total-variation of the distributional derivative,

$$\text{Per}(A_i) = \|D\chi_{A_i}\|_{\text{TV}} = \sup_{A_i} \left\{ \int \nabla \cdot \varphi d\lambda \dots (21) \right.$$

The generator of the chain acts on test functions ϕ by

$$\mathcal{L}\phi(A) = \int_S [\phi(B) - \phi(A)] K(A, dB) \dots (22)$$

The stationary distribution π satisfies the detailed balance equation

$$\pi(dB) = \int_S K(A, dB) \pi(dA) \dots (23)$$

and can equivalently be characterized as the minimizer over probability measures μ of a free-energy functional:

$$\pi = \arg \min_{\mu \in \mathcal{P}_2(S)} \mathbb{E}_{A \sim \mu} [E(A)] + 2\sigma^2 D_{\text{KL}}(\mu \| \mu_0) \dots (24)$$

where $\mathcal{P}_2(S)$ is the Wasserstein space of probability measures on S with finite second moment and D_{KL} is the Kullback-Leibler divergence.

A unifying Hilbert-space formulation identifies both approaches as seeking $f^* \in H$ minimizing a quadratic form $\langle Of, f \rangle_H$ over the fairness manifold $\mathcal{M}_{\text{EF}}$. For the mechanism-design model take

$$O = M = I - W \dots (25)$$

where the linear operator W acts componentwise by

$$(Wf)_i = \sum_{j=1}^n w_{ij} f_j, i = 1, \dots, n \dots (26)$$

and $W = (w_{ij})$ is doubly stochastic.

For the stochastic-geometric model take

$$O = -\mathcal{L} \dots (27)$$

where $\mathcal{L}$ denotes the self-adjoint Friedrichs extension (Birman, 1956; Lonigro *et al.*, 2019; Savin, 2024) of the pre-generator defined on $C_c^\infty(S)$.

Spectral comparison reduces to analyzing the eigenvalues $\lambda_k(M)$ and $\lambda_k(-L)$, ordered as

$$0 < \lambda_1 \leq \lambda_2 \leq \dots (28)$$

with associated eigen functions $\{\phi_k\} \subset H$ forming orthonormal bases for their respective invariant subspaces.

The fundamental question is whether there exists a bounded invertible operator

$$R = M^{-1/2} \mathcal{L} M^{-1/2} \dots (29)$$

with domain

$$\mathcal{D}(R) = \{f \in H : M^{-1/2} f \in \mathcal{D}(\mathcal{L})\} \dots (30)$$

such that there are positive constants c_1, c_2 satisfying the uniform spectral-comparison inequalities

$$c_1 \langle Mf, f \rangle_H \leq \langle -\mathcal{L}f, f \rangle_H \leq c_2 \langle Mf, f \rangle_H \text{ for all } f \in \mathcal{M}_{\text{EF}} \dots (31)$$

Establishing (31) gives spectral equivalence and permits direct comparison of convergence rates, bias resilience, and computational complexity for the two models within a single infinite-dimensional optimization framework over the measure space (Ω, F, λ) . From a rural-governance perspective, these properties correspond to three practical concerns: how quickly an allocation rule reaches a stable decision, how strongly it is affected by distorted or incomplete valuations, and whether it remains computationally feasible as the number of beneficiaries and the spatial heterogeneity of land increase.

Mechanism design as projection onto fairness subspaces

Let $W: H \rightarrow H$ denote the social welfare operator defined pointwise by

$$(Wf)_i(\omega) = \sum_{j=1}^n w_{ij} f_j(\omega) \dots (32)$$

where $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ is doubly stochastic, that is,

$$\sum_{i=1}^n w_{ij} = \sum_{j=1}^n w_{ij} = 1, w_{ij} \geq 0 \dots (33)$$

Assume W admits the spectral decomposition

$$W = \sum_{k=1}^n \mu_k u_k u_k^\top \dots (34)$$

where

$$1 = \mu_1 > \mu_2 \geq \dots \geq \mu_n \geq -1, u_1 = \frac{1}{\sqrt{n}} \mathbf{1} \dots (35)$$

The operator W extends to H by acting pointwise:

$$(Wf)(\omega) = Wf(\omega) \dots (36)$$

and hence

$$\|W\|_{\text{op}} = \mu_1 = 1, W^* = W^\top$$

with respect to $\langle \cdot, \cdot \rangle_H$.

Define

$$M = I - W \dots (38)$$

Then M is self-adjoint and positive semidefinite, with

$$\ker(M) = \text{span}\{\mathbf{1} \otimes g\} \dots (39)$$

and spectrum

$$\sigma(M) = \{1 - \mu_k\}_{k=1}^n = \{0\} \cup \{1 - \mu_k\}_{k=2}^n \dots (40)$$

The spectral gap is

$$\gamma(W) = 1 - \mu_2 > 0 \dots (41)$$

if and only if W is irreducible. The envy-free constraint manifold is

$$\mathcal{M}_{\text{EF}} = \{\chi_A\} \dots (42)$$

It is a closed subset of the unit simplex

$$\Sigma_H = \{g \in H\} \dots (43)$$

and it is proximal since H is uniformly convex. The metric projection

$$\Pi_{\mathcal{M}_{\text{EF}}} : H \rightarrow \mathcal{M}_{\text{EF}} \dots (44)$$

satisfies

$$\|f - \Pi_{\mathcal{M}_{\text{EF}}} f\|_H = \inf_{g \in \mathcal{M}_{\text{EF}}} \|f - g\|_H \dots (45)$$

and is single-valued by strict convexity of the norm. The mechanism design problem can be written as the variational inequality:

$$\begin{aligned} &\text{find } f^* \in \mathcal{M}_{\text{EF}} \text{ such that } \langle Mf^*, g - f^* \rangle_H \\ &\geq 0 \quad \forall g \in \mathcal{M}_{\text{EF}} \dots (46) \end{aligned}$$

Equivalently,

$$f^* = \Pi_{\mathcal{M}_{\text{EF}}}(Wf^*) \dots (47)$$

So f^* is a fixed point of the nonexpansive mapping $f \mapsto \Pi_{\mathcal{M}_{\text{EF}}}(Wf)$. By the Browder–Göhde–Kirk theorem,

since $\mathcal{M}_{EF}$ is closed, convex, and bounded in the weak topology and W is nonexpansive,

$$\text{Fix}(\Pi_{\mathcal{M}_{EF}} \circ W) \neq \emptyset \dots (48)$$

Strategy-proofness requires W to be firmly nonexpansive:

$$\|Wf - Wg\|_H^2 \leq \langle Wf - Wg, f - g \rangle_H \quad \forall f, g \in H \dots (49)$$

This holds if and only if $I - W$ is maximally monotone, meaning

$$\langle (I - W)f - (I - W)g, f - g \rangle_H \geq 0 \dots (50)$$

and the graph is not properly contained in the graph of any other monotone operator. Equivalently, W can be written as

$$W = \frac{1}{2}(I + N) \dots (51)$$

for some non-expansive N , or

$$W = J_A \dots (52)$$

for some maximally monotone A , where

$$J_A = (I + A)^{-1} \dots (53)$$

The Dubins-Spanier moving-knife algorithm (Walsh, 2011; Manabe & Okamoto, 2012; Ortega & Segal-Halevi, 2019) generalizes to the proximal point iteration

$$f^{(k+1)} = \Pi_{\mathcal{M}_{EF}} \left(Wf^{(k)} + \alpha_k \nabla U(f^{(k)}) \right) \dots (54)$$

where

$$U(f) = \sum_{i=1}^n \log(\langle f_i, \chi_{A_i} \rangle_{L^2}) \dots (55)$$

Its domain is

$$\text{dom}(U) = \{f : \dots\} \dots (56)$$

and its gradient is

$$(\nabla U(f))_i = \langle f_i, \chi_{A_i} \rangle \chi_{A_i} \dots (57)$$

The step sizes $\{\alpha_k\}$ satisfy

$$\sum_{k=0}^{\infty} \alpha_k = \infty, \sum_{k=0}^{\infty} \alpha_k^2 < \infty \dots (58)$$

the Robbins-Monro conditions. Convergence is analyzed via the Lyapunov function

$$V(f) = \frac{1}{2} \|f - f^*\|_H^2 \dots (59)$$

satisfying

$$V(f^{(k+1)}) \leq \left(1 - \frac{1}{1 + \|W\|_{op} \alpha_k \gamma(W)} \right) V(f^{(k)}) + 2\alpha_k^2 \|\nabla U(f^{(k)})\|_H^2 \dots (60)$$

This yields

$$\|f^{(k)} - f^*\|_H = O(k^{-1/2}) \dots (61)$$

for $\alpha_k \sim k^{-1}$, and linear convergence

$$\|f^{(k)} - f^*\|_H \leq (1 - 2\gamma(W))^k \|f^{(0)} - f^*\|_H \dots (62)$$

When U is strongly concave with modulus $\mu > 0$ and

$$\alpha_k \equiv \alpha < \frac{\mu + L^2}{2\gamma(W)} \dots (63)$$

where

$$L = \sup_{f \in \mathcal{M}_{EF}} \|\nabla^2 U(f)\|_{op} \dots (64)$$

The self-reducibility property of cake-cutting implies that for any $\varepsilon > 0$, an ε -envy-free partition can be computed in

$$O(n \log(n / \varepsilon)) \dots (65)$$

queries in the Robertson-Webb model, where agents report cut points

$$x = \inf \{y : \dots\} \dots (66)$$

d evaluation values

$$v_i([x, y]) = \int_x^y f_i d\lambda \dots (67)$$

The query-complexity lower bound is

$$\Omega(n \log(n / \varepsilon)) \dots (68)$$

by reduction from sorting. For the continuous Dubins-Spanier protocol on $[0, 1]$, the moving knife at position $x(t)$ satisfies

$$\frac{dx}{dt} = v(t) \dots (69)$$

where

$$v(t) = \sum_{i=1}^n f_i(x(t)) \dots (70)$$

and each agent i calls “stop” when

$$\int_0^{x(t)} f_i d\lambda = \frac{1}{n} \int_0^1 f_i d\lambda \dots (71)$$

This produces cut points

$$0 = c_0 < c_1 < \dots < c_n = 1 \dots (72)$$

with

$$A_i = [c_{i-1}, c_i) \dots (73)$$

and

$$\int_{A_i} f_i d\lambda = \frac{1}{n} v_i([0,1]) \dots (74)$$

guaranteeing proportionality; for identical valuations this yields exact equitability.

The Even-Paz divide-and-conquer protocol (Woeginger & Sgall, 2007) achieves

$$O(n \log n) \dots (75)$$

complexity via recursive bisection: given interval $[a, b]$ and agents $N' \subseteq N$, find $c \in [a, b]$ such that

$$\int_a^c f_i d\lambda = \frac{|N'|}{2} \int_a^b f_i d\lambda \dots (76)$$

for a median agent $i^* \in N'$, then recurse on

$$[a, c] \text{ with } N'_L = \left\{ i \in N' : \int_a^c f_i d\lambda \leq \frac{1}{2} \int_a^b f_i d\lambda \right\} \dots (77)$$

and on

$$[c, b] \text{ with } N'_R = N' \setminus N'_L \dots (78)$$

The Aziz-Mackenzie envy-free protocol (Aziz & Mackenzie, 2017) for $n = 3$ uses

$$O(n^{n^n}) \dots (79)$$

queries, demonstrating the super-exponential barrier for exact EF. The approximate EF protocol of Deng-Qi-Saberi achieves

$$(1 - \varepsilon) - EF \dots (80)$$

in

$$O(\varepsilon^{-n}) \dots (81)$$

queries for piecewise constant valuations. The mechanism design operator M admits the spectral decomposition

$$M = \sum_{k=2}^n (1 - \mu_k) P_k \dots (82)$$

Where P_k projects onto $u_k \otimes L^2(\Omega)$. The resolvent is

$$J_{\lambda M} = (I + \lambda M)^{-1} = \sum_{k=1}^n \frac{1}{1 + \lambda(1 - \mu_k)} P_k \dots (83)$$

With P_1 the projection onto constants. The Moreau envelope of the indicator $\iota_{\mathcal{M}_{EF}}$ is

$$\begin{aligned} e_{\lambda \iota_{\mathcal{M}_{EF}}}(f) &= \inf_{g \in \mathcal{M}_{EF}} \left\{ \iota_{\mathcal{M}_{EF}}(g) + \frac{1}{2\lambda} \|f - g\|_H^2 \right\} \\ &= \frac{1}{2\lambda} \|f - \Pi_{\mathcal{M}_{EF}} f\|_H^2 \dots (84) \end{aligned}$$

with proximal mapping

$$\text{prox}_{\lambda \iota_{\mathcal{M}_{EF}}} = \Pi_{\mathcal{M}_{EF}} v(85)$$

The Douglas–Rachford splitting for the composite problem

$$\min_{f \in H} U(f) + \iota_{\mathcal{M}_{EF}}(f) \dots (86)$$

yields the iteration

$$z^{(k+1)} = z^{(k)} + \Pi_{\mathcal{M}_{EF}} \left(2J_{\gamma M} z^{(k)} - z^{(k)} \right) - J_{\gamma M} z^{(k)} \dots (87)$$

which converges to z^* with

$$J_{\gamma M} z^* = f^* \dots (88)$$

The convergence rate is governed by the averaged-operator constant

$$\alpha = \frac{1}{2} \dots (89)$$

giving

$$\|z^{(k)} - z^*\|_H \leq \frac{1}{\sqrt{k}} \|z^{(0)} - z^*\|_H \dots (90)$$

for general monotone inclusions and

$$O((1 - 2\gamma(W))^k) \dots (91)$$

under strong monotonicity.

The incentive ratio

$$\rho = \sup_{f, f'} \frac{\langle f_i, \chi_{F_i(f)} \rangle}{\langle f_i, \chi_{F_i(f_i, f_{-i})} \rangle} \dots (92)$$

measures manipulation gain; for the proportional mechanism,

$$\rho = n \dots (93)$$

while for the randomized VCG mechanism,

$$\rho = 1 \dots (94)$$

though with

$$O(n^2) \dots (95)$$

communication complexity per agent.

The revelation principle states that for any mechanism (F, M) with equilibrium strategy profile s^* , there exists a direct mechanism (F', M') with

$$F'(f) = F(s^*(f)) \dots (96)$$

that is DSIC, reducing mechanism design to optimization over truthful social choice functions satisfying

$$f^* \in \arg \max_{g \in \mathcal{M}_{\text{EF}}} \Phi(f, g) \dots (97)$$

where

$$\Phi(f, g) = \sum_{i=1}^n \phi_i(\langle f_i, g_i \rangle_{L^2}) \dots (98)$$

and each ϕ_i is nondecreasing and concave.

The generalized VCG mechanism achieves this via

$$\Phi_{\text{VCG}}(f, g) = \sum_{i=1}^n f_i, g_i + \sum_{i=1}^n h_i(f_{-i}) \dots (99)$$

where

$$h_i(f_{-i}) = - \max_{g \in \mathcal{M}_{\text{EF}}} \sum_{j \neq i} f_j, g_j \dots (100)$$

ensuring

$$\langle f_i, \chi_{F_i(f)} \rangle + h_i(f_{-i}) \geq \langle f_i, \chi_{F_i(f', f_{-i})} \rangle + h_i(f_{-i}) \dots (101)$$

for all f'_i , thereby establishing DSIC by the taxation principle. The computational complexity of VCG is

$$P\text{-hard} \dots \#(102)$$

for general valuations, reducing from the partition problem. The price of anarchy for the proportional mechanism is

$$\text{PoA} = \frac{\mathbb{E}[\text{UW}(A_{\text{eq}})]}{\max_A \text{UW}(A)} \leq n \dots (103)$$

while for the Nash social welfare mechanism,

$$\text{PoA} \leq e^{1/e} \approx 1.445 \dots (104)$$

The α -fairness family

$$U_\alpha(A) = \sum_{i=1}^n \frac{u_i(A)^{1-\alpha}}{1-\alpha}, \alpha \in [0, \infty) \dots (105)$$

interpolates utilitarian fairness at $\alpha = 0$, Nash fairness as $\alpha \rightarrow 1$, and max-min fairness as $\alpha \rightarrow \infty$. The corresponding operator W_α has entries

$$w_{ij}^\alpha = \frac{u_j^{\alpha-1}}{\sum_{k=1}^n u_k^{\alpha-1}} \dots (106)$$

yielding

$$M_\alpha = I - W_\alpha \dots (107)$$

with spectral gap

$$\gamma_\alpha(W) = 1 - \mu_2^\alpha \dots (108)$$

where

$$\mu_2^\alpha = 1 - \frac{\sum_k u_k^{\alpha-1}}{n \min_i u_i^{\alpha-1}} \dots (109)$$

Approaches 1 as $\alpha \rightarrow \infty$, indicating vanishing convergence rate in the egalitarian limit.

The Birkhoff-von Neumann theorem (Paunescu & Radulescu, 2015; Marupanthorn, 2026) guarantees

$$W = \sum_{\sigma \in S_n} \theta_\sigma P_\sigma \dots (110)$$

Where P_σ are permutation matrices and

$$\theta_\sigma \geq 0, \sum_\sigma \theta_\sigma = 1 \dots (111)$$

This decomposes any doubly stochastic mechanism into deterministic allocations, with support size at most

$$n^2 - 2n + 2 \dots (112)$$

by the Carathéodory theorem. The Gale-Shapley stable (Gonczarowski *et al.*, 2018; Gokhale *et al.*, 2024; Hosseini *et al.*, 2024; Glitzner & Manlove, 2026) matching reduces to mechanism design when valuations are dichotomous:

$$f_i(\omega) \in \{0, 1\} \dots (113)$$

yielding the student-proposing deferred acceptance algorithm with complexity

$$O(nm) \dots (114)$$

For m land parcels. The top-trading-cycles mechanism achieves Pareto efficiency, individual rationality, and strategy-proofness for house allocation with complexity

$$O(n^2) \dots (115)$$

The pseudomarket mechanism of Hylland-Zeckhauser (He *et al.*, 2018) computes equilibrium prices

$$p: \Omega \rightarrow \mathbb{R}_{\geq 0} \dots (116)$$

and allocations

$$x_i : \Omega \rightarrow [0,1] \dots (117)$$

satisfying

$$\int_{\Omega} p(\omega) x_i(\omega) d\lambda = B_i, \dots (118)$$

$$\sum_{i=1}^n x_i(\omega) = 1 \lambda - a.e. \dots (119)$$

and

$$x_i(\omega) > 0 \Rightarrow \frac{f_i(\omega)}{p(\omega)} \geq \frac{f_j(\omega)}{p(\omega)} \quad \forall j \dots (120)$$

This yields a competitive equilibrium from equal incomes (CEEI) that is envy-free; existence follows from the Kakutani fixed-point theorem applied to the excess-demand correspondence

$$\zeta(p) = \sum_{i=1}^n x_i^*(p) - 1 \dots (121)$$

The computational complexity of CEEI is:

$$\text{PPAD-complete} \dots (122)$$

reducing from ε -Nash equilibrium computation.

The continuous-time limit of the discrete mechanism-design iteration is the differential inclusion

$$\frac{df}{dt} \in -Mf - N_{\mathcal{M}_{\text{EF}}}(f) \dots (123)$$

where

$$N_{\mathcal{M}_{\text{EF}}}(f) = \{v \in H : \langle v, g - f \rangle \leq 0 \dots (124)$$

is the normal cone. Its solutions satisfy

$$\frac{d}{dt} \|f(t) - f^*\|_H^2 \leq -2\gamma(W) \|f(t) - f^*\|_H^2 \dots (125)$$

by the strong monotonicity of M on $\mathcal{M}_{\text{EF}}^\perp$, yielding

$$\|f(t) - f^*\|_H \leq e^{-\gamma(W)t} \|f(0) - f^*\|_H \dots (126)$$

Stochastic-geometric models as Markov semigroups on $\mathbf{H}$

Let (S, d_H, π) denote the metric measure space with

$$S = \Delta_n(F) \hookrightarrow \mathbf{H} \text{ via } A \mapsto \chi_A \dots (127)$$

$$d_H(A, B) = \|\chi_A - \chi_B\|_H \dots (128)$$

and $\pi \in \mathcal{P}_2(S)$ the stationary distribution of the reversible Markov process $\{X_t\}_{t \geq 0}$ with càdlàg paths in S and transition kernel $p_t(A, dB)$ satisfying

$$p_{t+s}(A, \Gamma) = \int_S p_s(B, \Gamma) p_t(A, dB) \quad \forall \Gamma \in \mathcal{B}(S) \dots (129)$$

The Markov semigroup $\{P_t\}_{t \geq 0}$ acts on $L^2(S, \pi)$ by

$$P_t \varphi(A) = \int_S \varphi(B) p_t(A, dB) \dots (130)$$

with

$$P_0 = I, P_{t+s} = P_t P_s \dots (131)$$

and reversibility of π implies

$$\langle P_t \varphi, \psi \rangle_{L^2(\pi)} = \langle \varphi, P_t \psi \rangle_{L^2(\pi)} \dots (132)$$

The infinitesimal generator

$L : D(L) \subset L^2(S, \pi) \rightarrow L^2(S, \pi)$ is the closed self-adjoint negative semidefinite operator defined by

$$L\varphi = \lim_{t \downarrow 0} \frac{P_t \varphi - \varphi}{t} \dots (133)$$

so that

$$L1 = 0, \langle L\varphi, \varphi \rangle_{L^2(\pi)} \leq 0, \ker(L) = \text{span}\{1\} \dots (134)$$

The carré du champ is

$$\Gamma(\varphi, \psi) = \frac{1}{2} (L(\varphi\psi) - \varphi L\psi - \psi L\varphi) \dots (135)$$

with

$$\Gamma(\varphi) = \Gamma(\varphi, \varphi) \dots (136)$$

and the iterated carré du champ is

$$\Gamma_2(\varphi) = \frac{1}{2} L\Gamma(\varphi) - \Gamma(\varphi, L\varphi) \dots (137)$$

The Bakry-Émery curvature-dimension condition (Ambrosio *et al.*, 2015; Han & Sturm, 2019; Cushing *et al.*, 2020; Münch & Rose, 2020; Hu, 2026) $CD(\kappa, \infty)$ holds if

$$\Gamma_2(\varphi) \geq \kappa \Gamma(\varphi) \pi - a.e. \text{ for all } \varphi \in D(L^2) \dots (138)$$

implying

$$\lambda_1(-L) \geq \kappa \dots (139)$$

and the Poincaré inequality

$$\text{Var}_\pi(\varphi) \leq \frac{1}{\kappa} \int_S \Gamma(\varphi) d\pi \dots (140)$$

The energy functional $E : S \rightarrow \mathbb{R}_{\geq 0}$ is

$$E(A) = \sum_{i=1}^n \left(\int_{A_i} f_i d\lambda - \bar{u} \right)^2 + \lambda_{\text{reg}} \sum_{i=1}^n \text{Per}(A_i) \dots (141)$$

with

$$\bar{u} = \frac{1}{n} \sum_{j=1}^n \int_{A_j} f_j d\lambda \dots\dots (142)$$

and

$$\text{Per}(A_i) = \|D\chi_{A_i}\|_{\text{TV}} \dots\dots (143)$$

The stationary distribution is the Gibbs measure

$$\pi(dA) = Z^{-1} e^{-\beta E(A)} \mu_0(dA) \dots\dots (144)$$

where

$$Z = \int_S e^{-\beta E} d\mu_0 \dots\dots (145)$$

And μ_0 is the reference measure induced by the uniform distribution on $\Delta n(F)$. Equivalently,

$$\pi = \arg \min_{\mu \in \mathcal{P}_2(S)} \left\{ \mathbb{E}_\mu[E] + \beta^{-1} D_{\text{KL}}(\mu \mid \mu_0) \right\} \dots\dots (146)$$

The free energy $F : \mathcal{P}_2(S) \rightarrow \mathbb{R}$ is

$$F(\mu) = \int_S E d\mu + \beta^{-1} \int_S \log \left(\frac{d\mu}{d\mu_0} \right) d\mu \dots\dots (147)$$

and its Wasserstein gradient flow satisfies the Fokker–Planck equation

$$\partial_t \mu_t = \nabla_H \cdot \left(\mu_t \nabla_H \frac{\delta F}{\delta \mu}(\mu_t) \right) \dots\dots (148)$$

where

$$\frac{\delta F}{\delta \mu} = E + \beta^{-1} \log \left(\frac{d\mu}{d\mu_0} \right) \dots\dots (149)$$

and $\nabla_H \cdot$ is the divergence on H .

The Jordan-Kinderlehrer-Otto (JKO) scheme (Alvarez-Melis *et al.*, 2021; Candelieri *et al.*, 2024; Marino *et al.*, 2025; Wu & Wang, 2025; Rankin & Wong, 2026) discretizes this flow as

$$\mu_{k+1} = \arg \min_{\nu \in \mathcal{P}_2(S)} \left\{ \frac{1}{2\tau} W_2^2(\mu_k, \nu) + F(\nu) \right\} \dots\dots (150)$$

with $\tau > 0$. By the Brenier theorem, there exists a convex potential $\varphi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ such that the optimal map

$$T = \nabla_H \varphi \dots\dots (151)$$

pushes μ_k to μ_{k+1} , satisfying

$$T_{\#} \mu_k = \mu_{k+1} \dots\dots (152)$$

and

$$\frac{1}{2} W_2^2(\mu_k, \mu_{k+1}) = \int_H \|A - T(A)\|_H^2 d\mu_k(A) \dots\dots (153)$$

The Langevin dynamics on H is the SDE

$$dX_t = -\nabla_H E(X_t) dt + 2\beta^{-1/2} dW_t^H \dots\dots (154)$$

where W_t^H is a cylindrical Wiener process on H with covariance Q trace class, formally

$$W_t^H = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k \dots\dots (155)$$

for $\{e_k\}$ an orthonormal basis of H , $\lambda_k \geq 0$ with

$$\sum_{k=1}^{\infty} \lambda_k < \infty \dots\dots (156)$$

And $\{\beta_k\}$ independent standard Brownian motions.

The mild solution is

$$X_t = e^{tA} X_0 - \int_0^t e^{(t-s)A} \nabla_H E(X_s) ds + 2\beta^{-1/2} \int_0^t e^{(t-s)A} dW_s^H \dots\dots (157)$$

where

$$A = -\nu \Pi(-\Delta_H) \dots\dots (158)$$

with Π the Leray projector onto divergence-free fields in H and Δ_H the Laplacian.

The generator L admits the spectral decomposition

$$-L = \sum_{k=0}^{\infty} \lambda_k \langle \cdot, \psi_k \rangle_{L^2(\pi)} \psi_k \dots\dots (159)$$

with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\dots (160)$$

and $\{\psi_k\}$ an orthonormal basis of $L^2(S, \pi)$. The spectral gap $\lambda_1 > 0$ yields exponential convergence

$$\|P_t \varphi - \mathbb{E}_\pi[\varphi]\|_{L^2(\pi)} \leq e^{-\lambda_1 t} \|\varphi - \mathbb{E}_\pi[\varphi]\|_{L^2(\pi)} \dots\dots (161)$$

and the mixing time

$$\tau_{\text{mix}}(\varepsilon) = \inf \left\{ t > 0 : \sup_{A \in \mathcal{S}} \|p_t(A, \cdot) - \pi\|_{\text{TV}} \leq \varepsilon \right\} \dots\dots (162)$$

satisfies

$$\tau_{\text{mix}}(\varepsilon) \sim \lambda_1^{-1} \log(\varepsilon^{-1} \pi_{\min}^{-1}) \dots\dots (163)$$

where

$$\pi_{\min} = \inf_{A \in \mathcal{S}_0} \pi(\{A\}) \dots\dots (164)$$

for finite discretization $\mathcal{S}_0$.

Under $CD(\kappa, \infty)$, the log-Sobolev inequality holds:

$$\text{Ent}_\pi(\varphi^2) \leq \frac{2}{\kappa} \int_S \Gamma(\varphi) d\pi \dots\dots (165)$$

implying hypercontractivity

$$\|P_t\|_{L^p(\pi) \rightarrow L^q(\pi)} \leq 1, \dots (166)$$

for

$$t \geq \frac{1}{4\kappa} \log \left(\frac{q-1}{p-1} \right), 1 < p < q < \infty, \dots (167)$$

and the Talagrand transport inequality

$$W_2^2(\mu, \pi) \leq \frac{2}{\kappa} D_{\text{KL}}(\mu \mid \pi), \dots (168)$$

for all $\mu \in \mathcal{P}_2(S)$. The Dirichlet form is

$$\mathcal{E}_{\text{Dir}}(\varphi, \psi) = - \int_S \varphi L \psi \, d\pi = \int_S \Gamma(\varphi, \psi) \, d\pi, \dots (169)$$

with domain

$$\mathcal{D} = \left\{ \varphi \in L^2(\pi) : \mathcal{E}_{\text{Dir}}(\varphi, \varphi) < \infty \right\}, \dots (170)$$

The Metropolis–Hastings kernel (Tierney, 1998; Minh *et al.*, 2015; Andrieu *et al.*, 2020a, b; O’Leary & Wang, 2023; Guan & Jiao, 2025) on S is

$$K_{\text{MH}}(A, dB) = q(A, dB) \alpha(A, B) + \delta_A(dB) \left(1 - \int_S q(A, dC) \alpha(A, C) \right), \dots (171)$$

Where q is a symmetric proposal and

$$\alpha(A, B) = \min \left(1, e^{-\beta(E(B) - E(A))} \right), \dots (172)$$

The conductance is

$$\Phi_S = \inf_{0 < \pi(\cdot) \leq 1/2} \frac{\pi(\Gamma) \int_{\Gamma^c} K(A, \Gamma^c) \pi(dA)}{\pi(\Gamma)}, \dots (173)$$

and Cheeger’s inequality gives

$$2\Phi_S^2 \leq \lambda_1 \leq 2\Phi_S, \dots (174)$$

The unadjusted Langevin algorithm (ULA) discretizes the SDE as

$$A_{k+1} = A_k - \eta \nabla_H E(A_k) + 2\eta^{1/2} \beta^{-1/2} \xi_k, \dots (175)$$

with $\xi_k \sim N(0, Q)$ i.i.d. and step size

$$\eta < L^{-1}, L = \sup_{A \in S} \|\nabla_H^2 E(A)\|_{\text{op}}, \dots (176)$$

It yields weak error

$$\left| \mathbb{E}[\phi(A_K)] - \int_S \phi \, d\pi \right| \leq C_1 e^{-\lambda_1 \eta K} + C_2 \eta, \dots (177)$$

for $\phi \in C_b^4(S)$, and complexity

$$K \sim \varepsilon^{-2}, \dots (178)$$

to achieve accuracy ε .

The underdamped Langevin dynamics

$$dV_t = -\gamma V_t \, dt - \nabla_H E(X_t) \, dt + 2\gamma^{1/2} \beta^{-1/2} \, dW_t^H, \dots (179)$$

$$dX_t = V_t \, dt, \dots (180)$$

achieves accelerated convergence

$$\tilde{O}(\kappa), \dots (181)$$

where

$$\kappa = \frac{L}{\lambda_1}, \dots (182)$$

is the condition number.

The heat kernel is

$$p_t(A, B) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \psi_k(A) \psi_k(B), \dots (183)$$

and

$$\text{Tr}(P_t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}, \dots (184)$$

determines the spectral dimension

$$d_{\text{spec}} = 2 \lim_{t \rightarrow 0} \frac{\log(1/t)}{\log \text{Tr}(P_t)}, \dots (185)$$

The isoperimetric profile is

$$I_\pi(v) = \inf_{\pi(\Gamma)=v} \pi^+(\Gamma), \dots (186)$$

where

$$\pi^+(\Gamma) = \int_{\Gamma} K(A, \Gamma^c) \pi(dA), \dots (187)$$

Under $CD(\kappa, \infty)$, one has

$$I_\pi(v) \geq \kappa \sqrt{v \log(1/v)} \text{ for } v \leq \frac{1}{2}, \dots (188)$$

The coupling time

$$\tau_{\text{couple}} = \inf \{t \geq 0 : X_t = Y_t\}, \dots (189)$$

for reflection coupling satisfies

$$\mathbb{P}(\tau_{\text{couple}} > t) \leq e^{-\kappa t} \mathbb{E}[d_H(X_0, Y_0)], \dots (190)$$

by Kendall-Cranston, giving

$$\tau_{\text{mix}} \leq \kappa^{-1} \log \left(\frac{\varepsilon}{\text{diam}(S)} \right), \dots (191)$$

The coarse Ricci curvature is

$$\kappa_{\text{Oll}}(A, B) = 1 - \frac{W_1(K(A, \cdot), K(B, \cdot))}{d_H(A, B)} \dots (192)$$

and

$$\inf_{A \neq B} \kappa_{\text{Oll}}(A, B) \geq \kappa \dots (193)$$

implies

$$\lambda_1 \geq \kappa \dots (194)$$

and Gaussian concentration

$$\pi(\Gamma) \leq \exp(-2\kappa r^2) \dots (195)$$

for

$$\Gamma = \{A : d_H(A, A_0) \geq r\} \dots (196)$$

The empirical measure

$$\hat{\pi}^K = \frac{1}{K} \sum_{k=1}^K \delta_{A_k} \dots (197)$$

satisfies

$$\mathbb{E} \left[W_2^2(\hat{\pi}^K, \pi) \right] \leq \frac{C}{K} \int_S \int_0^\infty t p_t(A, A) dt \pi(dA) \dots (198)$$

by Fournier-Guillin.

The non-reversible perturbation

$$L_{\text{nr}} = L + B \dots (199)$$

With B antisymmetric in $L^2(\pi)$ and

$$B1 = 0 \dots (200)$$

Preserves π and reduces asymptotic variance:

$$\sigma_{\text{nr}}^2(f) = \lim_{T \rightarrow \infty} \frac{1}{T} \text{Var} \left(\int_0^T f(X_t^{\text{nr}}) dt \right) \leq \sigma^2(f) \dots (201)$$

The hypo-coercivity framework applies when

$$\Gamma_2(\varphi) + \Gamma_\perp(\varphi) \geq \lambda \|\varphi\|_H^2 \dots (202)$$

for a degenerate dissipation $\Gamma_\perp$, yielding exponential decay in $L^2(\pi)$ with rate

$$\lambda_{\text{eff}} > 0 \dots (203)$$

even if the symmetric spectral gap vanishes.

Spectral comparison of the two approaches

Let $M = I - W$ with

$$(Wf)_i = \sum_{j=1}^n w_{ij} f_j \dots (204)$$

and let $-L$ be the positive self-adjoint Friedrichs extension of the Markov generator on $L^2(S, \pi)$, both acting on dense domains in H and $L^2(S, \pi)$, respectively. The spectral decomposition of M is

$$M = \sum_{k=1}^\infty \lambda_k^M \langle \cdot, \phi_k^M \rangle_H \phi_k^M \dots (205)$$

with

$$0 = \lambda_0^M < \lambda_1^M \leq \lambda_2^M \leq \dots (206)$$

and $\{\phi_k^M\}$ orthonormal in H . For $k = 1, \dots, n$,

$$\lambda_k^M = 1 - \mu_k \dots (207)$$

Where μ_k are the eigenvalues of W , and for $k > n$,

$$\lambda_k^M = 1 \dots (208)$$

due to the infinite-dimensional fiber structure $H = L^2(\Omega; \mathbb{R}^n)$.

The spectral decomposition of $-L$ is

$$-L = \sum_{k=0}^\infty \lambda_k^L \langle \cdot, \psi_k \rangle_{L^2(\pi)} \psi_k \dots (209)$$

with

$$0 = \lambda_0^L < \lambda_1^L \leq \lambda_2^L \leq \dots (210)$$

and $\{\psi_k\}$ orthonormal in $L^2(S, \pi)$.

The relative spectral operator is

$$R = M^{-1/2} L M^{-1/2} \dots (211)$$

with domain

$$D(R) = \{f \in H \dots (212)$$

which is densely defined and closable. The spectral equivalence theorem states that if R is bounded with

$$0 < c_1 \leq R \leq c_2 < \infty \dots (213)$$

in the operator sense, then

$$c_1 \langle Mf, f \rangle_H \leq \langle -Lf, f \rangle_{L^2(\pi)} \leq c_2 \langle Mf, f \rangle_H \dots (214)$$

for all $f \in D(M) \cap D(L)$. The left inequality implies

$$\lambda_k^L \geq c_1 \lambda_k^M \dots (215)$$

and the right implies

$$\lambda_k^L \leq c_2 \lambda_k^M \dots (216)$$

for all k , by the Courant–Fischer min-max principle.

The constants c_1, c_2 are determined by

$$c_1 = \inf_{f \neq 0} \frac{\langle Mf, f \rangle}{\langle -Lf, f \rangle} \dots (217)$$

and

$$c_2 = \sup_{f \neq 0} \frac{\langle Mf, f \rangle}{\langle -Lf, f \rangle} \dots (218)$$

which are finite if and only if

$$\ker(M) \subseteq \ker(L) \dots (219)$$

and the ranges satisfy

$$R(M) \subseteq R(-L) \dots (220)$$

with closed complement. This holds when the fairness manifold $\mathcal{M}_{\text{EF}}$ is invariant under both dynamics and the stationary distribution π concentrates on $\mathcal{M}_{\text{EF}}$ with

$$\pi(\mathcal{M}_{\text{EF}}) = 1 \dots (221)$$

The spectral interlacing theorem states that if H_{EF} has codimension m in H and the active constraints define a closed subspace

$$V = \{f \in H : \dots\} \dots (222)$$

with

$$\dim V^\perp = m \dots (223)$$

then

$$\lambda_k^M \leq \lambda_{k+m}^L \leq \lambda_{k+2m}^M \dots (224)$$

for all $k \geq 1$. This follows from the Cauchy interlacing theorem applied to the compression

$$P_V(-L)|_V \dots (225)$$

Where P_V is the orthogonal projection onto V .

The condition numbers are

$$\kappa_{\text{MD}} = \frac{\lambda_{\min}(M|_{H_{\text{EF}}^\perp})}{\lambda_{\max}(M)} = \frac{1-\mu_2}{1-\mu_n} \dots (226)$$

for mechanism design and

$$\kappa_{\text{SG}} = \frac{\lambda_1^L}{\lambda_{\max}(-L)} \dots (227)$$

for stochastic-geometric. Under spectral equivalence,

$$\frac{c_2}{c_1} \kappa_{\text{MD}} \leq \kappa_{\text{SG}} \leq \frac{c_1}{c_2} \kappa_{\text{MD}} \dots (228)$$

Typically,

$$\kappa_{\text{SG}} \gg \kappa_{\text{MD}} \dots (229)$$

because

$$\lambda_{\max}(-L) \sim \tau^{-2n} \dots (230)$$

for discretization step τ , while

$$\lambda_1^L \sim e^{-\beta \Delta E} \dots (231)$$

for energy barrier

$$\Delta E = \max_{A, B \in \mathcal{S}} (E(A) - E(B)) \dots (232)$$

by Arrhenius scaling, whereas

$$\lambda_{\max}(M) = 1 - \mu_n \leq 2 \dots (233)$$

and

$$\lambda_{\min}(M|_{H_{\text{EF}}^\perp}) = 1 - \mu_2 \geq \frac{1}{n} \dots (234)$$

by the connectivity of the social network underlying W .

The convergence-rate comparison yields

$$\begin{aligned} \|f^{(k)} - f^*\|_H &\leq \left(1 - \frac{1}{1 + \|W\|_{\text{op}} \gamma(W)}\right)^k \|f^{(0)} - f^*\|_H \\ &= \left(1 - \frac{2}{1 - \mu_2}\right)^k \|f^{(0)} - f^*\|_H \dots (235) \end{aligned}$$

for mechanism design, versus

$$\|P_t \varphi - \mathbb{E}_\pi[\varphi]\|_{L^2(\pi)} \leq e^{-\lambda_1^L t} \|\varphi - \mathbb{E}_\pi[\varphi]\|_{L^2(\pi)} \dots (236)$$

for stochastic-geometric. Equating $t = k\Delta t$ with $\Delta t = \eta$, the ULA step size, gives

$$e^{-\lambda_1^L \eta k} \text{ versus } \left(1 - \frac{2}{1 - \mu_2}\right)^k \approx e^{-2(1-\mu_2)k} \dots (237)$$

so the ratio of exponential rates is

$$\frac{1 - \mu_2}{2\lambda_1^L \eta} \dots (238)$$

which is typically $\ll 1$ for $\eta = O(\lambda_1^L / L^2)$ by stability constraints, implying that the stochastic-geometric approach requires

$$O\left(\frac{\lambda_1^L \eta}{1 - \mu_2}\right) \sim O((\lambda_1^L)^2 L^2) \dots (239)$$

more iterations.

The Weyl law for the counting function

$$N_M(\lambda) = \{k : \lambda_k^M \leq \lambda\} \dots (240)$$

gives

$$N_M(\lambda) \sim \frac{(2\pi)^{-d} \omega_d |\Omega|}{\lambda^{d/2}} \dots (241)$$

for the Laplacian part on $\mathbb{R}^d$ with $d = 2$, while for $-L$ on the partition space S with metric dimension $d_H = \dim_{\text{Hausdorff}}(S)$,

$$N_L(\lambda) \sim C_S \lambda^{d_H/2} \dots (242)$$

with

$$d_H = n \cdot \dim_{\text{HB}}(\mathcal{F}) \dots (243)$$

Where $\dim_{\text{HB}}$ is the Hausdorff-Besicovitch dimension of the σ -algebra (Venieri, 2014; Katz & Zahl, 2018; Bakhtawar, 2020; Altaf *et al.*, 2023; Chen *et al.*, 2024; Michelucci, 2025). Since

$$d_H \gg 2 \dots (244)$$

the eigenvalue density

$$\frac{dN}{d\lambda} \sim \lambda^{d_H/2-1} \dots (245)$$

grows much faster for stochastic-geometric, implying smaller eigenvalue spacing and slower convergence in high-frequency modes. The heat trace asymptotics

$$\text{Tr}(e^{-tM}) \sim \sum_{j=0}^{\infty} a_j^M t^{(j-d)/2} \dots (246)$$

and

$$\text{Tr}(e^{tL}) \sim \sum_{j=0}^{\infty} a_j^L t^{(j-d_H)/2} \dots (247)$$

yield the spectral zeta functions

$$\zeta_M(s) = \text{Tr}(M^{-s}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \text{Tr}(e^{-tM}) dt \dots (248)$$

and

$$\zeta_L(s) = \text{Tr}((-L)^{-s}) \dots (249)$$

with meromorphic continuation to $\mathbb{C}$ and residues at

$$s = \frac{d}{2} \dots (250)$$

and

$$s = \frac{d_H}{2} \dots (251)$$

respectively, determining the spectral dimensions. The zeta-regularized determinants are

$$\det_{\zeta}(M) = \exp(-\zeta'_M(0)) \dots (252)$$

and

$$\det_{\zeta}(-L) = \exp(-\zeta'_L(0)) \dots (253)$$

appearing in the partition functions

$$Z_{\text{MD}} = (\det_{\zeta} M)^{-1/2} \dots (254)$$

and

$$Z_{\text{SG}} = (\det_{\zeta}(-L))^{-1/2} Z \dots (255)$$

Where Z is the Gibbs normalization.

The relative zeta function

$$\zeta_R(s) = \text{Tr}(R^{-s}) = \text{Tr}(M^{s/2}(-L)^{-s}M^{s/2}) \dots (256)$$

is well-defined for

$$\Re(s) > \frac{d_H}{2} \dots (257)$$

and its analytic continuation yields the functional equation relating

$$\zeta_R(s) \text{ to } \zeta_R(d_H - s) \dots (258)$$

The spectral projection operators

$$\Pi_M(\lambda) = 1_{(-\infty, \lambda]}(M) \dots (259)$$

and

$$\Pi_L(\lambda) = 1_{(-\infty, \lambda]}(-L) \dots (260)$$

have distribution functions

$$F_M(\lambda) = \text{Tr}(\Pi_M(\lambda)) \dots (261)$$

and

$$F_L(\lambda) = \text{Tr}(\Pi_L(\lambda)) \dots (262)$$

with densities

$$\rho_M(\lambda) = \frac{dF_M}{d\lambda} \dots (263)$$

and

$$\rho_L(\lambda) = \frac{dF_L}{d\lambda} \dots (264)$$

The spectral density of states is

$$\nu(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \text{ImTr}((M - \lambda - i\varepsilon)^{-1}) \dots (265)$$

for mechanism design, and similarly for L via the resolvent

$$(-L - \lambda - i\varepsilon)^{-1} \dots (266)$$

The Kreĭn spectral shift function is

where

$$S(z) = I - \frac{2i}{\text{Im}z} M^{1/2} (-L - z)^{-1} M^{1/2} \dots (268)$$

is the scattering matrix. It relates traces by

$$\text{Tr}(f(M) - f(-L)) = \int_{-\infty}^{\infty} f'(\lambda) \xi_{M,L}(\lambda) d\lambda \dots (269)$$

for sufficiently regular f , yielding the relative entropy

$$D_{\text{KL}}(\pi_M \parallel \pi_L) = \int_0^{\infty} \xi_{M,L}(\lambda) \log c_\lambda d\lambda \dots (270)$$

for appropriate normalization c_λ .

The trace-class property

$$M^{-1}L \in S_1(H) \dots (271)$$

implies

$$\sum_{k=1}^{\infty} \left| \frac{\lambda_k^M}{\lambda_k^L} - c \right| < \infty \dots (272)$$

for some $c > 0$, establishing asymptotic spectral equivalence at high frequencies.

The Szegő limit theorem for Toeplitz operators on H (Pérez-Esteva & Uribe, 2020; Camper, 2024; Miheisi & O'Loughlin, 2024; Montañó, 2025; Camper & Mitkovski, 2026) gives

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \varphi(\lambda_k^M) = \int_{\mathbb{T}^2} \varphi(\sigma(M)(\theta)) \frac{d\theta}{(2\pi)^2} \dots (273)$$

for continuous ϕ , where

$$\sigma(M)(\theta) = 1 - \frac{1}{n} \sum_{j=1}^n e^{i\theta_j} \dots (274)$$

is the symbol. Similarly for L with symbol

$$\sigma(-L)(\theta, \xi) = \beta |\nabla_H E(A)|^2 + \frac{1}{2} \sum_{i,j} Q_{ij} \xi_i \xi_j \dots (275)$$

in the semiclassical limit $\hbar \rightarrow 0$.

The quantum ergodicity theorem states that for quantum ergodic systems, the eigenfunctions ϕ_k^M equidistribute on the energy shell $\mathcal{M}_{EF}$ in the sense that

$$\lim_{k \rightarrow \infty} \phi_k^M, A \phi_k^M = \int_{\mathcal{M}_{EF}} A d\mu_{\text{Liouville}} \dots (276)$$

for pseudodifferential operators A of order zero, and similarly for ψ_k on S with respect to the Liouville measure induced by the Hamiltonian flow of E . This implies that high-frequency modes of both approaches become statistically indistinguishable, with local statistics governed by random matrix theory:

$$P(s) = \lim_{\lambda \rightarrow \infty} \frac{1}{N} \left\{ (i, j) : |\lambda_i - \lambda_j| \leq \frac{s}{\rho(\lambda)} \right\} \dots (277)$$

The nearest-neighbor spacing distribution converges to the Wigner-Dyson surmise

$$P_\beta(s) = 2\pi s e^{-\pi s^2/4} \dots (278)$$

For $\beta = 1$ or $\beta = 2$, depending on symmetry class.

The Berry–Tabor conjecture for integrable systems predicts

$$P(s) = e^{-s} \dots (279)$$

while the Bohigas–Giannoni–Schmit conjecture for chaotic systems predicts random-matrix statistics. The mechanism design operator M is integrable, being diagonalizable by Fourier transform on the torus when W is circulant, yielding

$$P_{\text{MD}}(s) = e^{-s} \dots (280)$$

while the stochastic-geometric operator $-L$ is chaotic, yielding

$$P_{\text{SG}}(s) = 2\pi s e^{-\pi s^2/4} \dots (281)$$

The transfer operator

$$L_{\text{transfer}} f(A) = \int_S e^{-\beta E(B)} f(B) \delta(B - \varphi(A)) dB \dots (282)$$

for the dynamical map $\phi: S \rightarrow S$ has spectrum related to $-L$ via

$$\lambda_k^{\text{transfer}} = e^{-\lambda_k^L t} \dots (283)$$

The Ruelle–Pollicott resonances are the poles of the meromorphic continuation of the dynamical zeta function

$$\zeta_{\text{Ruelle}}(z) = \exp \left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr}(L_{\text{transfer}}^n) \right) \dots (284)$$

which for expanding maps yields

$$\zeta_{\text{Ruelle}}(z) = \det(I - zL_{\text{transfer}})^{-1} \dots (285)$$

with zeros at

$$z_k = e^{\lambda_k^L} \dots (286)$$

The Selberg zeta function for the geodesic flow on S is

$$Z_{\text{Selberg}}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell(\gamma)}) \dots (287)$$

Where γ ranges over closed geodesics with length $\ell(\gamma)$, and its zeros determine the spectrum of $-L$ via the trace formula

$$\text{Tr}(e^{iL}) = \frac{4\pi t e^{-t/4}}{\sum_{\gamma} \frac{\sinh(\ell(\gamma)/2)}{\ell(\gamma)} e^{-\ell(\gamma)^2/4t}} + \sum_{k=0}^{\infty} e^{-\lambda_k^L t} \dots (288)$$

The Gutzwiller trace formula for the mechanism design operator is

$$\text{Tr}(e^{-itM}) = \sum_{\gamma} \frac{e^{iS_{\gamma}/\hbar - i\pi\mu_{\gamma}/2}}{|\det(I - M_{\gamma})|^{1/2} T_{\gamma}} + \sum_k e^{-i\lambda_k^M t} \dots (289)$$

Where γ are periodic orbits of the classical Hamiltonian $H = \sigma(M)$, T_{γ} their periods, M_{γ} the monodromy matrices, S_{γ} the actions, and μ_{γ} the Maslov indices. This semiclassical correspondence establishes that the spectral differences between the two approaches are rooted in their classical dynamical structures: integrable versus chaotic.

The Koopman–von Neumann operator

$$Kf = f \circ \varphi \dots (290)$$

on $L^2(S, \mu_{\text{Liouville}})$ is unitary with spectrum on the unit circle, and its generator

$$K_{\text{gen}} = \lim_{t \rightarrow 0} \frac{K_t - I}{t} \dots (291)$$

satisfies

$$-L = K_{\text{gen}}^* K_{\text{gen}} \dots (292)$$

when the noise is additive, linking the stochastic-geometric spectrum to the unitary Koopman spectrum via

$$\lambda_k^L = |\log z_k|^2 \dots (293)$$

for

$$z_k \in \sigma(K) \dots (294)$$

The comparison is completed by the spectral embedding theorem: there exists an isometric isomorphism

$$I : L^2(S, \pi) \rightarrow H_{\text{EF}}^{\perp} \dots (295)$$

such that

$$I(-L)I^{-1} = M|_{H_{\text{EF}}^{\perp}} \dots (296)$$

if and only if the spectral multiplicities match and the eigenfunctions satisfy

$$\psi_k = I^{-1} \phi_k^M \dots (297)$$

with

$$\|\psi_k\|_{L^2(\pi)} = \|\phi_k^M\|_H = 1 \dots (298)$$

This requires

$$\dim L^2(S, \pi) = \dim H_{\text{EF}}^{\perp} = \aleph_0 \dots (299)$$

and the existence of a unitary equivalence between the von Neumann algebras generated by the spectral projections

of $-L$ and $M|_{H_{\text{EF}}^{\perp}}$. This holds when the fairness constraints are non-degenerate and the land domain Ω has Lipschitz boundary.

Bias resilience via operator perturbation theory

Let $B: H \rightarrow H$ be the bias operator, with $\|B\|_{\text{op}} = 1$, and suppose either $B^* = B$ (self-adjoint bias) or $B^* = -B$ (antisymmetric drift bias), modeling systematic distortions in agent valuations or spatial allocation. Define the perturbed operators

$$M_{\varepsilon} = M + \varepsilon B \dots (300)$$

and

$$L_{\varepsilon} = L + \varepsilon B \dots (301)$$

for $\varepsilon \in [0, \varepsilon_0]$, where $\varepsilon_0 < \min(\gamma(W)^{-1}, \lambda_1^{-1})$ ensures stability. By the Kato-Rellich theorem (Schmid & Griesemer, 2014; Much, 2015; Ito & Skibsted, 2016; Humaikani *et al.*, 2025), since M is self-adjoint with compact resolvent and B is relatively bounded with bound $\alpha < 1$, meaning

$$\|Bf\|_H \leq a \|Mf\|_H + b \|f\|_H \quad \forall f \in D(M) \dots (302)$$

the perturbed operator M_{ε} is self-adjoint on $D(M)$.

Its isolated eigenvalues admit the expansion

$$\lambda_k^M(\varepsilon) = \lambda_k^M + \varepsilon \lambda_k^{(1)} + \varepsilon^2 \lambda_k^{(2)} + O(\varepsilon^3) \dots (303)$$

where

$$\lambda_k^{(1)} = \langle \phi_k^M, B \phi_k^M \rangle_H \dots (304)$$

and

$$\lambda_k^{(2)} = \sum_{j \neq k} \frac{|\langle \phi_j^M, B \phi_k^M \rangle_H|^2}{\lambda_k^M - \lambda_j^M} \dots (305)$$

The eigen-projectors expand as

$$P_k(\varepsilon) = P_k + \varepsilon P_k^{(1)} + \varepsilon^2 P_k^{(2)} + O(\varepsilon^3) \dots (306)$$

with

$$P_k = \langle \cdot, \phi_k^M \rangle_H \phi_k^M \dots (307)$$

$$P_k^{(1)} = -P_k B S_k - S_k B P_k \dots (308)$$

$$P_k^{(2)} = S_k B P_k B S_k - S_k B S_k B P_k - P_k B S_k B S_k + S_k^{(2)} B P_k B P_k + P_k B P_k B S_k^{(2)} \dots (309)$$

$$S_k = \sum_{j \neq k} \frac{1}{\lambda_k^M - \lambda_j^M} P_j \dots (310)$$

and

$$S_k^{(2)} = \sum_{j \neq k} \frac{1}{(\lambda_k^M - \lambda_j^M)^2} P_j \dots (311)$$

The perturbed fairness solution is

$$f_\varepsilon^* = f^* + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + O(\varepsilon^3) \dots (312)$$

with

$$f^{(1)} = S_1 B f^* = \sum_{j \neq 1} \frac{\langle \phi_j^M, B f^* \rangle_H}{\lambda_1^M - \lambda_j^M} \phi_j^M \dots (313)$$

and

$$f^{(2)} = S_1 \left(B - \lambda_1^{(1)} \right) f^{(1)} - \lambda_1^{(2)} S_1^2 B f^* \dots (314)$$

The fairness deviation satisfies

$$\delta_{MD}(\varepsilon) = \|f_\varepsilon^* - f^*\|_H = \varepsilon \|S_1 B f^*\|_H + \varepsilon^2 \|S_1 (B S_1 B - \langle f^*, B S_1 B f^* \rangle_H) f^*\|_H + O(\varepsilon^3) \dots (315)$$

and

$$\delta_{MD}(\varepsilon) \leq \gamma(W)^{-1} \varepsilon \|B f^*\|_H + \gamma(W)^{-2} \varepsilon^2 \|B\|_{op}^2 \|f^*\|_H + O(\varepsilon^3) \dots (316)$$

since

$$\|S_1\|_{op} = \frac{1}{\lambda_2^M - \lambda_1^M} = \gamma(W)^{-1} \dots (317)$$

and

$$\|B f^*\|_H \leq \|f^*\|_H \dots (318)$$

For the stochastic-geometric model, the perturbed generator $L_\varepsilon = L + \varepsilon B$ has stationary distribution π_ε satisfying

$$L_\varepsilon^* \pi_\varepsilon = 0 \dots (319)$$

By the Donsker-Varadhan large deviation principle, the rate function for the empirical measure

$$\hat{\pi}_T = \frac{1}{T} \int_0^T \delta_{X_t} dt \dots (320)$$

is

$$I(\mu) = -\inf_{u>0} \int_S \frac{L u}{u} d\mu = \frac{1}{4} \int_S \left| \nabla \log \left(\frac{d\mu}{d\pi_\varepsilon} \right) \right|^2 d\pi_\varepsilon \dots (321)$$

for $\mu \ll \pi_\varepsilon$.

The relative density expands as

$$\frac{d\pi_0}{d\pi_\varepsilon} = 1 + \varepsilon \psi + \varepsilon^2 \psi^{(2)} + O(\varepsilon^3) \dots (322)$$

where the first-order correction ψ solves the Poisson equation

$$-L\psi = B^* 1 - \mathbb{E}_{\pi_0} [B^* 1], \mathbb{E}_{\pi_0} [\psi] = 0 \dots (323)$$

and

$$\|\psi\|_{L^2(\pi_0)} \leq \lambda_1^{-1} \|B^* 1 - \mathbb{E}_{\pi_0} [B^* 1]\|_{L^2(\pi_0)} \dots (324)$$

by the Poincaré inequality. The second-order correction satisfies

$$-L\psi^{(2)} = B\psi - \mathbb{E}_{\pi_0} [B\psi] - \psi \mathbb{E}_{\pi_0} [B^* 1] \dots (325)$$

with

$$\|\psi^{(2)}\|_{L^2(\pi_0)} \leq \lambda_1^{-1} (\|B\|_{op} \|\psi\|_{L^2(\pi_0)} + \|\psi\|_{L^2(\pi_0)}) \|B^* 1\|_{L^2(\pi_0)} \dots (326)$$

The total variation bias is

$$\|\pi_\varepsilon - \pi_0\|_{TV} = \frac{1}{2} \int_S \left| \frac{d\pi_0}{d\pi_\varepsilon} - 1 \right| d\pi_0 = 2\varepsilon \int_S |\psi| d\pi_0 + 2\varepsilon^2 \int_S |\psi^{(2)}| d\pi_0 + O(\varepsilon^3) \dots (327)$$

and

$$\|\pi_\varepsilon - \pi_0\|_{TV} \leq 2\lambda_1^{-1/2} \varepsilon \|B^* 1\|_{L^2(\pi_0)} + 2\lambda_1^{-3/2} \varepsilon^2 \|B\|_{op} \|B^* 1\|_{L^2(\pi_0)} + O(\varepsilon^3)$$

The comparative bias-resilience theorem states that if B is local, meaning

$$(Bf)_i = b_i f_i \text{ with } b_i \in L^\infty(\Omega), \|b_i\|_{L^\infty} \leq 1 \dots (329)$$

then

$$\delta_{MD}(\varepsilon) \sim \gamma(W)^{-1} \varepsilon \left(\sum_{j \neq 1} (\lambda_j^M - \lambda_1^M)^2 |\langle \phi_j^M, B f^* \rangle|^2 \right)^{1/2} \dots (330)$$

while

$$\|\pi_\varepsilon - \pi_0\|_{TV} \sim \lambda_1^{-1} \varepsilon \text{Var}_{\pi_0} (B^* 1)^{1/2} \dots (331)$$

Since $\lambda_1 \ll \gamma(W)$ in typical parameter regimes where the Markov chain explores high-energy barriers, the ratio satisfies

$$\frac{\|\pi_\varepsilon - \pi_0\|_{TV}}{\delta_{MD}(\varepsilon)} \sim \frac{\gamma(W)}{\lambda_1} \frac{\text{Var}_{\pi_0} (B^* 1)^{1/2}}{\|S_1 B f^*\|_H} \ll 1 \dots (332)$$

establishing that mechanism design is more resilient to local valuation perturbations, whereas the stochastic-geometric approach is more resilient to global spatial distortions due to the ergodic averaging

$$\frac{1}{T} \int_0^T B(X_t) dt \rightarrow \mathbb{E}_{\pi_0} [B] \dots (333)$$

at rate $O(T^{-1/2})$ by the central limit theorem for Markov processes with asymptotic variance

$$\sigma_{\text{asym}}^2(B) = 2 \int_0^\infty \langle P_t B, B \rangle_{L^2(\pi_0)} dt = 2 \langle (-L)^{-1} B, B \rangle_{L^2(\pi_0)} \leq \lambda_1^{-2} \|B\|_{L^2(\pi_0)}^2 \dots (334)$$

The bias-variance tradeoff operator is

$$E_{\text{MSE}}(\varepsilon) = \varepsilon^2 B^* B + \sigma^2 (-L)^{-1} \dots (335)$$

acting on $L^2(\pi_0)$, with eigenvalues

$$\nu_k(\varepsilon) = \varepsilon^2 \beta_k^2 + \sigma^2 \lambda_k^{-1} \dots (336)$$

Where β_k are the singular values of B . The optimal bias magnitude minimizing the trace is

$$\varepsilon^* = \sigma \left(\sum_{k=1}^\infty \beta_k^2 \right)^{-1/2} \left(\sum_{k=1}^\infty \lambda_k^{-1} \right)^{1/2} \dots (337)$$

yielding

$$\min \text{Tr}(E_{\text{MSE}}) = 2\sigma \left(\sum_k \beta_k^2 \right)^{1/2} \left(\sum_k \lambda_k^{-1} \right)^{1/2} \dots (338)$$

For the mechanism-design side, the regularized bias operator is

$$B_{\text{reg}} = (M + \tau I)^{-1} B \dots (339)$$

with regularization parameter $\tau > 0$, and the robust fairness solution is

$$f_{\varepsilon, \tau}^* = \arg \min_{f \in H_{\text{EF}}} \left\{ \langle Mf, f \rangle + \varepsilon \langle Bf, f \rangle + \tau \|f\|_H^2 \right\} = (M + \varepsilon B + \tau I)^{-1} Mf^* \dots (340)$$

with error

$$\|f_{\varepsilon, \tau}^* - f^*\|_H \leq \frac{\varepsilon + \tau}{\gamma(W) + \tau} \|f^*\|_H \dots (341)$$

The Feshbach-Schur map for singular perturbations when ε crosses a spectral threshold is

$$F_M(z) = PBP - PBQ(QMQ - z)^{-1}QBP \dots (342)$$

with $P = P_k$ and $Q = I - P$. The effective Hamiltonian

$$M_{\text{eff}}(z) = PMP + \varepsilon F_M(z) \dots (343)$$

has eigenvalues determined by the implicit equation

$$z = \lambda_k^M + \varepsilon \phi_k, B\phi_k - \varepsilon^2 \sum_{j \neq k} \frac{|\langle \phi_j, B\phi_k \rangle|^2}{\lambda_j^M - z} + O(\varepsilon^3) \dots (344)$$

The spectral shift function for the pair (M, M^ε) is

$$\xi_{M, \varepsilon}(\lambda) = \frac{1}{\pi} \lim_{\delta \rightarrow 0^+} \arg \det(I + \varepsilon B(M - \lambda - i\delta)^{-1}) \dots (345)$$

and

$$\text{Tr}(f(M_\varepsilon) - f(M)) = \int_{-\infty}^\infty f'(\lambda) \xi_{M, \varepsilon}(\lambda) d\lambda \dots (346)$$

for $f \in C_0^2(\mathbb{R})$.

For the stochastic-geometric model, the perturbed spectral gap is

$$\lambda_1(\varepsilon) = \lambda_1 + \varepsilon \lambda_1^{(1)} + \varepsilon^2 \lambda_1^{(2)} + O(\varepsilon^3) \dots (347)$$

with

$$\lambda_1^{(1)} = \langle \psi_1, B\psi_1 \rangle_{L^2(\pi_0)} \dots (348)$$

and

$$\lambda_1^{(2)} = \sum_{k=2}^\infty \frac{|\langle \psi_k, B\psi_1 \rangle|^2}{\lambda_1 - \lambda_k} \dots (349)$$

by analytic perturbation theory for isolated eigenvalues. The perturbed eigenfunction is

$$\psi_1(\varepsilon) = \psi_1 + \varepsilon \sum_{k=2}^\infty \frac{\langle \psi_k, B\psi_1 \rangle}{\lambda_1 - \lambda_k} \psi_k + O(\varepsilon^2) \dots (350)$$

yielding the perturbed mixing time

$$\tau_{\text{mix}}(\varepsilon) = \lambda_1(\varepsilon)^{-1} \log(\pi_{\text{min}}^{-1} \varepsilon^{-1}) \sim \lambda_1^{-1} \left(1 - \frac{\lambda_1^{(1)}}{\lambda_1} \varepsilon \right) \log(\pi_{\text{min}}^{-1} \varepsilon^{-1}) \dots (351)$$

The Lyapunov-Schmidt reduction for the nonlinear fixed-point equation

$$f = \Pi_{\mathcal{M}_{\text{EF}}}(Wf + \varepsilon Bf) \dots (352)$$

decomposes as

$$Pf = P\Pi_{\mathcal{M}_{\text{EF}}}(WPf + WQf + \varepsilon PBPf + \varepsilon PBQf) \dots (353)$$

and

$$Qf = Q\Pi_{\mathcal{M}_{\text{EF}}}(WPf + WQf + \varepsilon QBPf + \varepsilon QBQf) \dots (354)$$

Solving the Q -equation by the implicit function theorem yields

$$Qf = h(Pf, \varepsilon) \dots (355)$$

with

$$h(0, 0) = 0, D_1 h(0, 0) = 0 \dots (356)$$

Substituting into the P -equation gives the bifurcation equation

$$F(Pf, \varepsilon) = P\Pi_{\mathcal{M}_{EF}} \begin{pmatrix} WPf + Wh(Pf, \varepsilon) \\ +\varepsilon PBPf + \varepsilon PBh(Pf, \varepsilon) \end{pmatrix} \\ -Pf = 0, \dots (357)$$

whose Jacobian

$$D_1 F(0, 0) = PMP, \dots (358)$$

is singular on $\ker(M)$. This requires the Crandall–Rabinowitz transversality condition

$$\phi_1^M, D_1 \partial_\varepsilon F(0, 0) \phi_1^M \neq 0, \dots (359)$$

for a pitchfork bifurcation of fairness solutions. The robustness radius is

$$r_{\text{rob}} = \inf \{ \varepsilon > 0 : \exists f \in \mathcal{M}_{EF}, \} = \frac{\gamma(W)}{\|B\|_{\text{op}}} \dots (360)$$

For the stochastic-geometric model, the robustness radius is

$$r_{\text{rob}}^{\text{SG}} = \inf \{ \varepsilon > 0 : \lambda_1(\varepsilon) = 0 \} = \frac{\lambda_1}{\|B\|_{\text{op}}} \dots (361)$$

by Weyl's bound

$$\lambda_1(\varepsilon) \geq \lambda_1 - \varepsilon \|B\|_{\text{op}}, \dots (362)$$

Since

$$\lambda_1 \ll \gamma(W), \dots (363)$$

it follows that

$$r_{\text{rob}}^{\text{SG}} \ll r_{\text{rob}}^{\text{MD}}, \dots (364)$$

confirming the comparative fragility of the Markov chain to bias-induced spectral collapse.

The higher-order cumulants of the bias response are

$$\kappa_m^{\text{MD}} = \langle f^*, (BS_1)^m f^* \rangle_H, \dots (365)$$

and

$$\kappa_m^{\text{SG}} = \mathbb{E}_{\pi_0} \left[(B(-L)^{-1})^m 1 \right], \dots (366)$$

with generating functions

$$G_{\text{MD}}(z) = \langle f^*, (I - zBS_1)^{-1} f^* \rangle = \sum_{m=0}^{\infty} z^m \kappa_m^{\text{MD}}, \dots (367)$$

and

$$G_{\text{SG}}(z) = \mathbb{E}_{\pi_0} \left[(I - zB(-L)^{-1})^{-1} 1 \right], \dots (368)$$

convergent for

$$|z| < \gamma(W), \dots (369)$$

and

$$|z| < \lambda_1, \dots (370)$$

respectively.

Computational complexity and algorithmic implementations

Let $J_{\alpha_k M}$ denote the resolvent used in the proximal point algorithm for mechanism design. The iteration is

$$f^{(k+1)} = \arg \min_{g \in \mathcal{M}_{EF}} \left\{ \langle Wf^{(k)}, g \rangle_H + \frac{1}{2\alpha_k} \|g - f^{(k)}\|_H^2 \right\} \\ = J_{\alpha_k M} \left(f^{(k)} \right), \dots (371)$$

where

$$J_{\alpha M} = (I + \alpha M)^{-1}, \dots (372)$$

The resolvent admits the spectral representation

$$J_{\alpha M} f = \sum_{k=1}^{\infty} \frac{1}{1 + \alpha \lambda_k^M} \langle f, \phi_k^M \rangle_H \phi_k^M, \dots (373)$$

with $\{\phi_k^M\}$ the eigenbasis of M and

$$\lambda_k^M = 1 - \mu_k, \dots (374)$$

The Moreau envelope of the indicator $\iota_{\mathcal{M}_{EF}}$ is

$$e_{\lambda \iota_{\mathcal{M}_{EF}}}(f) = \frac{1}{2\lambda} \|f - \Pi_{\mathcal{M}_{EF}} f\|_H^2, \dots (375)$$

with proximal mapping

$$\text{prox}_{\lambda \iota_{\mathcal{M}_{EF}}} = \Pi_{\mathcal{M}_{EF}}, \dots (376)$$

The Douglas-Rachford splitting for

$$\min_{f \in H} U(f) + \iota_{\mathcal{M}_{EF}}(f), \dots (377)$$

iterates

$$z^{(k+1)} = z^{(k)} + \Pi_{\mathcal{M}_{EF}} \left(2J_{\gamma M} z^{(k)} - z^{(k)} \right) - J_{\gamma M} z^{(k)}, \dots (378)$$

converging to z^* with

$$J_{\gamma M} z^* = f^*, \dots (379)$$

Under strong monotonicity,

$$\|z^{(k)} - z^*\|_H \leq (1 - 2\gamma(W))^k \|z^{(0)} - z^*\|_H, \dots (380)$$

while for general maximal monotone inclusions,

$$\|z^{(k)} - z^*\|_H = O(k^{-1/2}), \dots (381)$$

The alternating direction method of multipliers reformulates the same problem as

$$\min_{f, g} \{ U(f) + \iota_{\mathcal{M}_{EF}}(g) : f - g = 0 \}, \dots (382)$$

Its augmented Lagrangian is

$$L_\rho(f, g, \lambda) = U(f) + \iota_{\mathcal{M}_{\text{EF}}}(g) + \langle \lambda, f - g \rangle_H + \frac{\rho}{2} \|f - g\|_H^2 \dots (383)$$

The updates are

$$f^{(k+1)} = \arg \min_f \left\{ U(f) + \langle \lambda^{(k)}, f \rangle + \frac{\rho}{2} \|f - g^{(k)}\|_H^2 \right\} \\ = (\nabla U + \rho I)^{-1}(\rho g^{(k)} - \lambda^{(k)}) \dots (384)$$

$$g^{(k+1)} = \Pi_{\mathcal{M}_{\text{EF}}} \left(f^{(k+1)} + \rho^{-1} \lambda^{(k)} \right) \dots (385)$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \rho \left(f^{(k+1)} - g^{(k+1)} \right) \dots (386)$$

The iteration complexity is

$$O(\kappa_{\text{MD}} \log(1/\varepsilon)) \dots (387)$$

to reach

$$\|f^{(k)} - f^*\|_H \leq \varepsilon \dots (388)$$

where

$$\kappa_{\text{MD}} = \frac{1 - \mu_2}{1 - \mu_n} \dots (389)$$

The discrete Dubins-Spanier protocol on $[0, 1]$ computes cut points

$$c_i = \inf \left\{ x : \int_0^x f_i \, d\lambda \geq \frac{1}{n} \int_0^1 f_i \, d\lambda \right\} \dots (390)$$

with total complexity

$$O(n^2) \dots (391)$$

The Even-Paz divide-and-conquer protocol achieves

$$O(n \log n) \dots (392)$$

queries via the recurrence

$$T(n) = 2T(n/2) + O(n) \dots (393)$$

The Aziz-Mackenzie exact envy-free protocol for $n = 3$ uses

$$O(n^n) \dots (394)$$

queries, establishing the super-exponential lower barrier. The Deng-Qi-Saberi $(1-\varepsilon)$ -EF protocol for piecewise constant valuations achieves

$$O(n/\varepsilon) \dots (395)$$

query complexity. The Robertson-Webb query model lower bound is

$$\Omega(n \log(n/\varepsilon)) \dots (396)$$

For ε -EF by reduction from sorting.

The VCG mechanism requires computing

$$\max_{A \in \Delta_n(\mathcal{F})} \sum_{i=1}^n f_i, \chi_{A_i} \dots (397)$$

which is $\#P$ -hard for general valuations, by reduction from the number partition problem. The Hylland-Zeckhauser CEEI computation is

$$\text{PPAD-complete} \dots (398)$$

by reduction from ε -Nash equilibrium.

The stochastic-geometric unadjusted Langevin algorithm is

$$A_{k+1} = A_k - \eta \nabla_H E(A_k) + 2\eta^{1/2} \beta^{-1/2} \xi_k \dots (399)$$

with

$$\xi_k \sim N(0, Q) \text{ i.i.d.} \dots (400)$$

and step-size constraint

$$\eta < L^{-1} \dots (401)$$

where

$$L = \sup_{A \in \mathcal{S}} \|\nabla_H^2 E(A)\|_{\text{op}} \dots (402)$$

The weak error satisfies

$$\left| \mathbb{E}[\phi(A_K)] - \int_S \phi \, d\pi \right| \leq C_1 e^{-\lambda \eta K} + C_2 \eta \dots (403)$$

for $\phi \in C_b^4(S)$, yielding optimal complexity

$$K = \tilde{O}(\varepsilon^{-2}) \dots (404)$$

to achieve

$$W_2(\mathcal{L}(A_K), \pi) \leq \varepsilon \dots (405)$$

with

$$\eta \sim \varepsilon \dots (406)$$

The underdamped Langevin dynamics is

$$dV_t = -\gamma V_t \, dt - \nabla_H E(X_t) \, dt + 2\gamma^{1/2} \beta^{-1/2} dW_t^H \dots (407)$$

$$dX_t = V_t \, dt \dots (408)$$

and achieves accelerated convergence

$$\tilde{O}(\kappa) \dots (409)$$

where

$$\kappa = \frac{L}{\lambda_1} \dots (410)$$

The Metropolis-adjusted Langevin algorithm accepts proposals with probability

$$\alpha(A_k, A_{k+1}) = \min\left(1, \frac{\pi(A_k)q(A_k, A_{k+1})}{\pi(A_{k+1})q(A_{k+1}, A_k)}\right) \dots (411)$$

where

$$q(A, B) \propto \exp\left(-\frac{1}{4\eta} \|B - A + \eta \nabla E(A)\|_H^2\right) \dots (412)$$

It has asymptotic acceptance rate

$$0.574 \dots (413)$$

which is optimal in the diffusion limit.

The JKO scheme discretizes the Fokker-Planck gradient flow as

$$\mu_{k+1} = \arg \min_{\nu \in \mathcal{P}_2(S)} \left\{ \frac{1}{2\tau} W_2^2(\mu_k, \nu) + F(\nu) \right\} \dots (414)$$

with

$$F(\nu) = \int_S E d\nu + \beta^{-1} D_{\text{KL}}(\nu \| \mu_0) \dots (415)$$

By Brenier's theorem,

$$\mu_{k+1} = T_{\#} \mu_k \dots (416)$$

for

$$T = \nabla_H \varphi \dots (417)$$

With ϕ convex. Each step requires solving the Monge–Ampère equation

$$\det(\nabla_H^2 \varphi) = \frac{\mu_k}{\mu_{k+1} \circ \nabla_H \varphi} \dots (418)$$

with complexity

$$O(m^3) \dots (419)$$

For m -sample discretization via Sinkhorn iteration

$$K_{ij} = \exp\left(-\frac{1}{\varepsilon} d_H^2(A_i, B_j)\right) \dots (420)$$

$$u^{(\ell+1)} = K v^{(\ell)} \odot a \dots (421)$$

$$v^{(\ell+1)} = K^\top u^{(\ell+1)} \odot b \dots (422)$$

converging linearly at rate

$$\kappa(K) \dots (423)$$

The empirical measure

$$\hat{\pi}^K = \frac{1}{K} \sum_{k=1}^K \delta_{A_k} \dots (424)$$

satisfies

$$\mathbb{E}[W_2^2(\hat{\pi}^K, \pi)] \leq CK^{-1/d_H} \dots (425)$$

for

$$d_H > 2 \dots (426)$$

by Fournier-Guillin.

The quantum phase estimation algorithm for mechanism design computes the ground state f^* with query complexity

$$O(\gamma(W)^{-1} \log(1/\varepsilon)) \dots (427)$$

to precision ε , versus classical power iteration

$$O(\gamma(W)^{-2} \log(1/\varepsilon)) \dots (428)$$

yielding quadratic speedup. The quantum walk operator

$$W_{\text{walk}} = S(2\Pi - I) \dots (429)$$

with shift S and reflection Π about the stationary subspace has spectral gap δ , giving mixing time

$$O(\delta^{-1}) \dots (430)$$

versus the same classical scaling up to constant factors in the discrete model. The quantum linear systems algorithm solves

$$Mf = b \dots (431)$$

with condition number

$$\kappa_{\text{MD}} \dots (432)$$

in

$$O(\kappa_{\text{MD}}^2 \log(n)/\varepsilon) \dots (433)$$

queries versus classical

$$O(\kappa_{\text{MD}} n^2) \dots (434)$$

For stochastic-geometric methods, quantum simulated annealing achieves

$$O(E_{\max} / \sqrt{\lambda_1}) \dots (435)$$

versus classical

$$O(E_{\max} / \lambda_1) \dots (436)$$

for energy barrier $E_{\max}$, with the quantum advantage depending on the implementation and the availability of coherent oracles.

The complexity comparison is summarized by

$$\tau_{\text{MD}}^{\text{classical}} = O(\gamma(W)^{-1} \kappa_{\text{MD}} \log(\varepsilon^{-1})) \dots (437)$$

$$\tau_{MD}^{\text{quantum}} = O\left(\gamma(W)^{-1} \kappa_{MD}^{1/2} \log(\varepsilon^{-1})\right) \dots (438)$$

$$\tau_{SG}^{\text{classical}} = O\left(\lambda_1^{-1} \kappa_{SG} \log(\varepsilon^{-1})\right) \dots (439)$$

$$\tau_{SG}^{\text{quantum}} = O\left(\lambda_1^{-1} \kappa_{SG}^{1/2} \log(\varepsilon^{-1})\right) \dots (440)$$

The space complexity is

$$S_{MD} = O(n|\Omega|) \dots (441)$$

for mechanism design versus

$$S_{SG} = O(|\Omega|^n) \dots (442)$$

for stochastic-geometric state space. The communication complexity of the VCG mechanism is

$$C_{VCG} = O(n^2 |\mathcal{F}|) \dots (443)$$

bits per round, while the Langevin sampler requires

$$C_{SG} = O(n|\Omega| \log |\Omega|) \dots (444)$$

bits per iteration.

The parallel depth is

$$D_{MD} = O(\log n) \dots (445)$$

for the Even–Paz tree versus

$$D_{SG} = O(1) \dots (446)$$

for independent Langevin chains. The arithmetic circuit complexity of evaluating $E(A)$ is

$$AC_0 \dots (447)$$

for piecewise constant valuations and

$$\text{P-complete} \dots (448)$$

for general L^2 densities.

The sample complexity for learning the valuation density f_i from m samples is

$$m = O\left(\varepsilon^{-2} \log(\delta^{-1})\right) \dots (449)$$

for uniform convergence by VC-dimension

$$d_{VC} = O(n) \dots (450)$$

The generalization error for the empirical mechanism $\hat{F}^m$ satisfies

$$\mathbb{E}[\|\hat{F}^m - F\|_{TV}] \leq C m^{-1/2} d_{VC} \dots (451)$$

The total computational cost for ε -optimal fairness is

$$C_{\text{total}}^{\text{MD}} = O\left(\varepsilon^{-2} n^2 |\Omega| \log(\varepsilon^{-1})\right) \dots (452)$$

and

$$C_{\text{total}}^{\text{SG}} = O\left(\lambda_1^{-1} \varepsilon^{-2} |\Omega|^n \log(\varepsilon^{-1})\right) \dots (453)$$

This establishes an exponential separation in dimension n favoring mechanism design for large communities, but favoring stochastic-geometric methods for spatially heterogeneous domains where $|\Omega|$ dominates. In applied terms, the result suggests that the choice of allocation procedure should depend on the institutional and spatial context rather than on a single universal rule. Direct fair-division mechanisms are especially relevant where many beneficiaries must be served transparently, whereas stochastic exploration may be useful when substantial variation in land characteristics makes spatial uncertainty a central concern.

Conclusion and perspectives

The unified Hilbert-space framework shows that mechanism design with fair-division algorithms and stochastic-geometric Markov models provide two mathematically distinct but comparable ways of addressing equitable public-land allocation. The contribution is relevant to community agriculture because the allocation of a productive resource such as land is not determined by efficiency alone: it also involves heterogeneous preferences, distributive fairness, institutional credibility and the capacity of an allocation procedure to remain robust under uncertainty. The spectral formulation makes these dimensions comparable without replacing the social and institutional judgement required in real land-governance settings.

$$H = L^2(\Omega, \lambda; \mathbb{R}^n) \dots (454)$$

In particular, the social welfare operator is

$$M = I - W \dots (455)$$

with spectrum

$$\sigma(M) = \{1 - \mu_k\}_{k=1}^n \cup \{1\} \dots (456)$$

and spectral gap

$$\gamma(W) = 1 - \mu_2 \dots (457)$$

The negative Markov generator is

$$-L \dots (458)$$

with spectrum

$$\sigma(-L) = \{\lambda_k^L\}_{k=0}^\infty \dots (459)$$

and spectral gap

$$\lambda_1 > 0 \dots (460)$$

Both approaches seek minimizers of

$$\langle Of, f \rangle_H \dots (461)$$

over the envy-free constraint manifold

$$\mathcal{M}_{\text{EF}} \subset H \dots (462)$$

For mechanism design, the solution is

$$f^* = \Pi_{\mathcal{M}_{\text{EF}}} (Wf^*) \dots (463)$$

obtained via the firmly nonexpansive resolvent iteration

$$f^{(k+1)} = J_{\alpha_k M} (f^{(k)}) \dots (464)$$

converging at rate

$$\|f^{(k)} - f^*\|_H \leq (1 - 2\gamma(W))^k \|f^{(0)} - f^*\|_H \dots (465)$$

For the stochastic-geometric model, the stationary distribution is

$$\pi = \arg \min_{\mu \in \mathcal{P}_2(S)} \{ \mathbb{E}_\mu[E] + \beta^{-1} D_{\text{KL}}(\mu \| \mu_0) \} \dots (466)$$

obtained via the Langevin dynamics

$$dX_t = -\nabla_H E(X_t) dt + 2\beta^{-1/2} dW_t^H \dots (467)$$

with mixing time

$$\tau_{\text{mix}} \sim \lambda_1^{-1} \log(\varepsilon^{-1} \pi_{\min}^{-1}) \dots (468)$$

The spectral equivalence theorem, via the relative operator

$$R = M^{-1/2} L M^{-1/2} \dots (469)$$

establishes

$$c_1 \langle Mf, f \rangle_H \leq \langle -Lf, f \rangle_{L^2(\pi)} \leq c_2 \langle Mf, f \rangle_H \dots (470)$$

It also yields the interlacing relation

$$\lambda_k^M \leq \lambda_{k+m}^L \leq \lambda_{k+2m}^M \dots (471)$$

for active-constraint codimension m .

The comparison therefore reveals a context-dependent trade-off

$$\gamma(W)^{-1} \lambda_1^{-1} = \frac{\mathbb{E}_\pi[E]}{2\text{Var}_\pi(E)} \exp(k_B T \Delta F) \dots (472)$$

which shows that mechanism design dominates when $\gamma(W)$ is large and valuations are truthful, while stochastic-geometric methods dominate when spatial heterogeneity requires exploration governed by. For community agriculture, this distinction has a clear decision-making interpretation: procedures based on reported preferences are most credible when valuation information can be elicited transparently, whereas stochastic approaches become more attractive when uncertainty is strongly linked to the spatial characteristics of the land itself.

$$\lambda_1 \ll \gamma(W) \dots (473)$$

The bias-resilience analysis shows

$$\delta_{\text{MD}}(\varepsilon) \sim \gamma(W)^{-1} \varepsilon \dots (474)$$

whereas

$$\|\pi_\varepsilon - \pi_0\|_{\text{TV}} \sim \lambda_1^{-1} \varepsilon \dots (475)$$

Hence

$$\frac{\|\pi_\varepsilon - \pi_0\|_{\text{TV}}}{\delta_{\text{MD}}(\varepsilon)} \sim \frac{\gamma(W)}{\lambda_1} \ll 1 \dots (476)$$

confirming mechanism design's superior resilience to local perturbations and stochastic-geometric methods' superior resilience to global distortions via ergodic averaging. This distinction matters for rural institutions because local bias may arise from inaccurate or strategic valuation by particular beneficiaries, while global distortion may reflect broader spatial heterogeneity affecting the land domain. The framework therefore separates two sources of uncertainty that require different governance responses.

The computational complexity comparison yields

$$C_{\text{MD}} = O(\varepsilon^{-2} n^2 |\Omega| \log(\varepsilon^{-1})) \dots (477)$$

versus

$$C_{\text{SG}} = O(\lambda_1^{-1} \varepsilon^{-2} |\Omega|^n \log(\varepsilon^{-1})) \dots (478)$$

The corresponding quantum speedups are

$$O(\gamma(W)^{-1} \kappa_{\text{MD}}^{1/2} \log(\varepsilon^{-1})) \dots (479)$$

and

$$O(\lambda_1^{-1} \kappa_{\text{SG}}^{1/2} \log(\varepsilon^{-1})) \dots (480)$$

The spectral hybrid operator

$$H_\alpha = (1 - \alpha)M + \alpha(-L) \dots (481)$$

with $\alpha \in [0, 1]$, interpolates both approaches with effective spectral gap

$$\gamma_\alpha = (1 - \alpha)\gamma(W) + \alpha\lambda_1 + \alpha(1 - \alpha)\text{Gap}_{\text{cross}} \dots (482)$$

where

$$\text{Gap}_{\text{cross}} = \langle M\phi_1^L, \phi_1^L \rangle - \langle L\phi_1^M, \phi_1^M \rangle \dots (483)$$

The spectral hybrid operator provides a potential compromise for community agriculture where equity, robustness and computational feasibility must coexist. Its value is primarily as a decision-support concept: it shows how the relative weight given to direct fairness rules and stochastic adaptation can be adjusted to the characteristics of the community and the land. Future empirical work should test these theoretical results using observed parcel characteristics and beneficiary valuations, and should

examine how tenure arrangements, participation, gendered access to productive resources, institutional trust and livelihood dependence on agriculture influence the practical acceptability of alternative allocation procedures. The open problem of spectral-gap scaling on fractal domains with Hausdorff dimension d_H conjectures

$$\lambda_1 \sim n^{-2/d_H} |\Omega|^{-2/d_H} \dots (484)$$

while the quantum-classical separation problem asks whether mechanism design lies in $BQP \setminus BPP$ relative to an oracle. The noncommutative extension replaces H with

$$L^2(\mathcal{M}, \tau) \dots (485)$$

for a von Neumann algebra $\mathcal{M}$, to model entangled fairness constraints

$$\rho_{ij} = \text{Tr} \left(\rho_{\text{global}} (E_i \otimes E_j) \right) \neq \rho_i \otimes \rho_j \dots (486)$$

The final synthesis is that spectral geometry provides a rigorous mathematical language for studying fairness in land partitioning, but the practical meaning of fairness remains rooted in the way communities govern access to a scarce productive resource. Both approaches can be interpreted as projections onto constraint subspaces governed by self-adjoint operators, while the choice between them depends on the structure of preferences, the spatial heterogeneity of land, the type of uncertainty and the institutional objective pursued. In this sense, the mathematical results complement rather than replace socio-economic judgement in public-land governance.

$$(\gamma, \lambda_1, \kappa, \tau_{\text{mix}}) \dots (487)$$

Thus the geometry of fairness is precisely the spectral theory of justice, expressed through

$$\min_{\alpha \in [0,1]} \max \left\{ \gamma(W)^{1-\alpha}, \lambda_1^\alpha, \gamma_\alpha^2 \alpha(1-\alpha) \|\text{Gap}_{\text{cross}}\| \right\} \dots (488)$$

subject to

$$f_\alpha^* \in \mathcal{M}_{\text{EF}} \dots (489)$$

and

$$\pi_\alpha(\mathcal{M}_{\text{EF}}) = 1 \dots (490)$$

which completes the mathematical formulation of equitable public-land partitioning within the unified Hilbert-space framework. The framework should therefore be read as an interdisciplinary contribution: advanced mathematical tools are used to clarify a problem of agricultural resource allocation, while field validation remains necessary before operational recommendations are made for specific rural communities or public land programmes.

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