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Review Article

The socioeconomics of groundwater depletion and technology adoption: a mean field game and stochastic dynamics framework for rural agricultural systems

Claudette Dickmi Vaillam*, Farida Adama, Brice Simo, Bertrand Zing Zing

Institut de Recherche Agricole pour le Développement (IRAD), P.O. Box 2123, Yaoundé, Cameroun

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*Corresponding Author: Claudette Dickmi Vaillam, E-mail: claudettedick@yahoo.com

Abstract

This article develops a comprehensive mathematical framework for the scenario analysis of coupled human-natural systems in rural agricultural contexts, integrating stochastic partial differential equations (SPDEs), mean field games (MFG), Pontryagin optimal control, dynamical systems bifurcation theory, and Bayesian nonlinear filtering into a unified analytical architecture. The system is formulated as a coupled stochastic dynamical system on a spatiotemporal domain, where a continuum of smallholder farmers strategically extract from a common-pool aquifer, adopt climate-resilient technologies through spatially nonlocal social learning, and optimize intertemporal profits under incomplete climate information. We derive the mean-field Nash equilibrium via the coupled Hamilton-Jacobi-Bellman and Fokker-Planck equations, characterize the optimal switching boundary for binary technology adoption decisions, and prove existence and uniqueness of variational solutions for the groundwater and adoption SPDEs under uniform ellipticity and Lipschitz conditions. The deterministic skeleton of the coupled system is analyzed for Hopf, saddle-node, and cusp bifurcations, revealing critical thresholds for regime shifts and hysteresis in the groundwater-adoption state space. Bayesian learning is modeled through the Kushner-Stratonovich and Zakai equations, yielding adaptive expectations and a quantifiable learning premium. Scenario ensemble analysis employs quasi-Monte Carlo methods, Sobol variance decomposition, and polynomial chaos expansions to rank policy sensitivities and compute risk-adjusted outcomes. Six high-dimensional visualizations: surface plots, wireframe plots, mesh plots, and volume plots, illustrate the spatiotemporal groundwater depletion, crop yield response surfaces, Nash payoff manifolds, technology diffusion fronts, and climate-adaptation risk volumes. Policy implications include dynamically optimal groundwater pricing, adoption subsidies internalizing social learning externalities, climate information infrastructure investment, and robust decision-making under deep uncertainty.

Keywords: Adaptive expectations, Bifurcation analysis, Common-pool resources, Mean field games, Stochastic partial differential equation

Introduction

Problem statement and mathematical framework

We consider a bounded spatial domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, over a finite temporal horizon $[0,T]$ where $T < \infty$ (Löscher *et al*., 2026). The coupled human-natural system is characterized by a state vector $z: \Omega \times [0,T] \to \mathbb{R}^6$ defined as:

$$z(x,t) = \begin{bmatrix} G(x,t) \\ A(x,t) \\ S(x,t) \\ L(x,t) \\ m(x,t) \\ v(x,t) \end{bmatrix} \in Z \quad \ldots\ldots (1)$$

and

$$Z \subseteq H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times P(\Omega) \times C^{2,1}(\Omega \times [0,T]) \quad \ldots\ldots (2)$$

where G denotes groundwater depth, A denotes technology adoption rate, S denotes soil organic carbon density, L denotes livestock stocking density, m denotes the population distribution in the mean-field game, and v denotes the value function. The control vector $u: \Omega \times [0,T] \to U \subset \mathbb{R}^4$ is given by:

$$u(x,t) = \begin{bmatrix} q(x,t) \\ F(x,t) \\ I(x,t) \\ \dot{A}_{\text{ctrl}}(x,t) \end{bmatrix} \ldots\ldots (3)$$

where q is groundwater extraction rate, F is fertilizer application, I is the binary adoption decision, and $\dot{A}_{\text{ctrl}}$ is the controlled adoption rate.

The master coupled stochastic dynamical system

The complete evolution of the system is governed by the following coupled stochastic partial differential equation (SPDE) system (Dai, 2024):

$$dz = F(z,u,\mu;\theta)dt + \Sigma(z) \cdot dW_t \ldots\ldots (4)$$

where $F: Z \times U \times M \times \Theta \to Z$ is the deterministic drift operator, $\Sigma: Z \to L(H;Z)$ is the diffusion operator, $\mu \in M \subset \mathbb{R}^p$ represents policy parameters, $\theta \in \Theta \subset \mathbb{R}^q$ represents climate state variables, and $W_t = (W_t^G, W_t^A, W_t^S, W_t^L, W_t^m, W_t^v)^T$ is a cylindrical Wiener process on a separable Hilbert space $H = L^2(\Omega)$ with covariance operator $Q \in L_1^+(H)$ satisfying:

$$\mathbb{E}\left[dW_t^i(x) dW_s^j(y)\right] = \delta(t-s)Q_{ij}(x,y)dt\, ds, \forall i,j \in \{G,A,S,L,m,v\} \ldots\ldots (5)$$

The covariance kernel is specified as a Matérn-3/2 spatial correlation:

$$Q_{ij}(x,y) = \sigma_i\sigma_j\rho_{ij}\big(1 + \ell_{ij}^3\|x - y\|\big)\exp\big(-\ell_{ij}^3\|x-y\|\big) \quad \ldots\ldots(6)$$

where $\rho_{ij} \in [-1,1]$ are cross-correlation coefficients and $\ell_{ij} > 0$ are characteristic length scales.

Component-wise system decomposition

Groundwater hydrology (SPDE)

The groundwater component evolves according to the stochastic nonlinear porous media equation (Bernard *et al*., 2025):

$$dG = \left[\nabla\cdot(K(G)\nabla G) - \sum_{k=1}^{N_p} Q_k(t)\delta(x-x_k) + R(G,\theta)\right]dt + \sigma_G(G)dW_t^G \quad \ldots\ldots(7)$$

where the nonlinear hydraulic conductivity follows:

$$K(G) = K_0\exp(\alpha G) + K_1\left(\frac{G_{\text{sat}}}{G}\right)^{\beta_G}, \alpha, \beta_G > 0 \quad \ldots\ldots(8)$$

The stochastic recharge function $R(G,\theta)$ incorporates climate uncertainty:

$$R(G,\theta) = n_e R_{\text{eff}}(\theta)\left(1-\frac{G}{G_{\max}}\right)^{\gamma_R} - E(G,T_{\text{air}}) \ldots\ldots(9)$$

with effective rainfall $R_{\text{eff}}(\theta) = \theta_1 + \theta_2\sin(3652\pi t) + \sigma_R\xi_t$, where $\xi_t \sim N(0,1)$ is a white noise process, and evapotranspiration:

$$E(G,T_{\text{air}}) = E_0\big(1+\kappa_T(T_{\text{air}}-T_{\text{ref}})\big)\left(\frac{G_{\text{crit}}}{G_{\text{crit}}-G}\right) + \delta_E \ldots\ldots(10)$$

where $(\cdot)_+ = \max(\cdot,0)$.

Technology adoption (Cross-diffusion SPDE)

The adoption rate $A \in [0,1]$ satisfies a generalized Fisher-KPP equation with cross-diffusion and nonlocal social learning (Fuentealba & Quaas, 2021):

$$dA = [\nabla\cdot(D_A(A,G)\nabla A + J_{\text{cross}}(A,G)) + F_A(A,G) + N(A)]dt + \sigma_A(A)dW_t^A \ldots\ldots(11)$$

where the diffusion tensor is:

$$D_A(A,G) = D_0\Big(I + \beta_1 AI + \beta_2\frac{G_{\text{ref}}}{G}I + \beta_3(\nabla A\otimes\nabla A)\Big)\ldots\ldots(12)$$

The cross-diffusion flux is:

$$J_{\text{cross}}(A,G) = \chi_A\nabla G + \chi_2 A(1-A)\nabla S \ldots\ldots(13)$$

The local reaction term is:

$$F_A(A,G) = r_A A\left(1-\frac{A}{K_A}\right)\left(\frac{G_{\text{crit}}}{G_{\text{crit}}-G}\right) + \nu_A\left(1+\eta_A\frac{S_{\text{ref}}}{S}\right)\ldots\ldots(14)$$

The nonlocal social learning kernel is:

$$N(A)(x,t) = \int_\Omega \kappa(x,y)[A(y,t)-A(x,t)]dy \ldots\ldots(15)$$

$$\kappa(x,y) = \kappa_0\exp(-2\sigma_{\text{soc}}^2\|x-y\|^2)\Big(1 + \lambda_{\text{soc}}\mathbf{1}_{\{\|x-y\|\le d_{\text{village}}\}}\Big) \ldots\ldots(16)$$

Soil organic carbon dynamics

$$dS = [D_S\nabla^2 S + I_S(F,L,A) - D_S(S,T_{\text{soil}})]dt + \sigma_S(S)dW_t^S \ldots\ldots(17)$$

with carbon input:

$$I_S(F,L,A) = \phi_F\frac{1+\psi_F F}{F} + \phi_L L + \phi_A A\left(1+\frac{S_{\text{sat}}}{S}\right) - \omega_S \ldots\ldots(18)$$

and decomposition:

$$D_S(S,T_{\text{soil}}) = k_{\text{dec}}S\exp\left(-\frac{R_g T_{\text{soil}}}{E_a}\right)\left(1+\frac{G_{\text{opt}}}{G}\right)^{\zeta s} \ldots\ldots(19)$$

Livestock carrying capacity

$$dL = \left[r_L L\left(1-\frac{L}{K_L(G,S)}\right) + \nabla\cdot(D_L\nabla L)\right]dt + \sigma_L L\,dW_t^L \ldots\ldots(20)$$

where the groundwater-dependent carrying capacity is:

$$K_L(G,S) = K_{L,0}\left(\frac{G_{\text{ref}}}{G}\right)^{\theta_L}\left(1+\frac{S_{\text{ref}}}{S}\right)^{\phi_L}\exp\big(-2\sigma_G^2(G-G_{\text{opt}})^2\big)\ldots\ldots(21)$$

Mean-field population distribution (Fokker-Planck)

The population distribution $m \in P(\Omega)$ evolves via the forward Kolmogorov equation (Bratus & Rozanova, 2026):

$$\partial_t m = -\nabla\cdot\big(mf(z,u^*,\bar{u})\big) + \frac{1}{2}\nabla\cdot\nabla\cdot\big(ma(z)\big) + L_{\text{FP}}^* m \ldots\ldots(22)$$

where f is the deterministic drift of the representative agent, $a = \Sigma Q\Sigma^T$ is the diffusion matrix, and $\bar{u}(t) = \int_\Omega u^*(x,t)m(x,t)\,dx$ is the mean-field control.

Value function (HJB Equation)

The value function $v(x,t)$ satisfies the Hamilton-Jacobi-Bellman equation (Tang *et al*., 2025):

$$\partial_t v + \frac{1}{2}\mathrm{tr}(a(z)\nabla^2 v) + \sup_{u\in U}\{e^{-rt}\Pi(z,u) + \nabla v \cdot f(z,u,\bar{u})\} = 0 \ldots\ldots (23)$$

with terminal condition:

$$v(x,T) = e^{-rT}\Phi\big(z(x,T)\big) \ldots\ldots (24)$$

where the instantaneous profit function is:

$$\Pi(z,u) = p_Y Y(R,F,G,A) + p_M M(L,G) - c_q(q,G) - c_F(F) - c_I I - c_A \dot{A}_{\mathrm{ctrl}} - w(t)q \ldots\ldots (25)$$

with crop yield:

$$Y(R,F,G,A) = A_0[\alpha_Y(\delta R - \rho + (1-\delta)F - \rho) - \rho^{\gamma} + \beta_Y G - \gamma] - \gamma_1(1+\varepsilon_Y) \ldots\ldots (26)$$

and livestock output:

$$M(L,G) = M_0 L^{\alpha_M}\left(\frac{G_{\mathrm{ref}}}{G}\right)^{\beta_M}\exp\left(-\frac{G}{\lambda_M L^2}\right) \ldots\ldots (27)$$

Functional spaces and regularity

We define the state space:

$$Z = \{z \in H:\} \ldots\ldots (28)$$

where $H = H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)\times L^{\infty}(\Omega)\times L^1(\Omega)\times W^{2,p}(\Omega)$. The control space is:

$$U = \{u \in \mathbb{R}^4:\} \ldots\ldots (29)$$

Initial and boundary conditions

Initial conditions:

$$z(x,0) = z_0(x) = \begin{bmatrix} G_0(x) \\ A_0(x) \\ S_0(x) \\ L_0(x) \\ m_0(x) \\ v_0(x) \end{bmatrix} \ldots\ldots (30)$$

with compatibility conditions:

$$\int_{\Omega} m_0(x)dx = 1, m_0(x) \geq 0, \mathrm{supp}(m_0) \subseteq \Omega \ldots\ldots (31)$$

Boundary conditions on $\partial\Omega = \Gamma_D \cup \Gamma_N$:

$$G|_{\Gamma_D} = G_D, n\cdot(K(G)\nabla G)|_{\Gamma_N} = 0 \ldots\ldots (32)$$

$$n\cdot(D_A\nabla A + J_{\mathrm{cross}})|_{\partial\Omega} = 0, n\cdot(D_S\nabla S)|_{\partial\Omega} = 0 \ldots\ldots (33)$$

$$n\cdot(D_L\nabla L)|_{\partial\Omega} = 0, n\cdot\left(mf - \frac{1}{2}\nabla\cdot(ma)\right)|_{\partial\Omega} = 0 \ldots\ldots (34)$$

Parameter space and assumptions

The complete parameter vector is:

$$\Theta = (K_0, K_1, \alpha, \beta_G, D_0, \beta_1, \beta_2, \beta_3, \chi, \chi_2, r_A, K_A, \nu_A, \eta_A, \kappa_0, \sigma_{\mathrm{soc}}, \lambda_{\mathrm{soc}}, \ldots\ w, r, A_0, \alpha_Y, \delta, \rho, \gamma, \beta_Y, M_0, \alpha_M, \beta_M, \lambda_M)^T \in \mathbb{R}^{45} \ldots\ldots \#(35)$$

(i) Assumption 1.1. (Regularity): $K \in C^2(\mathbb{R}_+)$, $D_A \in C^1([0,1]\times\mathbb{R}_+;\mathbb{R}^{2\times 2})$, $\Pi \in C^{2,1}(Z\times U)$, and $\Phi \in C^2(Z)$.

(ii) Assumption 1.2. (Coercivity): There exists $\lambda > 0$ such that for all $\xi \in \mathbb{R}^2$:

$$\xi^T D_A(A,G)\xi \geq \lambda|\xi|^2, \forall (A,G) \in [0,1]\times\mathbb{R}_+ \ldots\ldots (36)$$

(iii) Assumption 1.3. (Mean-Field Consistency): The mean-field control satisfies:

$$\bar{q}^*(t) = \int_{\Omega} q^*\big(G(x,t), \nabla v(x,t)\big)m(x,t)dx \ldots\ldots (37)$$

and the master equation closure:

$$\partial_t U + H(z,\nabla U,m) + \int_{\Omega}\frac{\delta m}{\delta U}(x,m,y)L^*_{\mathrm{FP}}m(y)dy = 0 \ldots\ldots (38)$$

where $U(t,z,m)$ is the master value function.

Well-posedness and existence

Theorem 1.1. (Well-posedness): Under Assumptions 1.1 to 1.3, with $z_0 \in Z$ and $u \in L^{\infty}(0,T;U)$, there exists a unique mild solution $z \in C([0,T];H)\cap L^2(0,T;V)$ to the coupled system (1.2), where:

$$V = H^2(\Omega)\times H^2(\Omega)\times H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)\times H^2(\Omega) \ldots\ldots (39)$$

Proof sketch: The proof proceeds via a fxed-point argument on the Banach space $C([0,T];H)$ (Naudts, 2026). Define the solution operator $S: H \to C([0,T];H)$ mapping initial data to the mild solution of the linearized system. Using the Lipschitz continuity of F and the boundedness of Σ on bounded subsets of Z (Mateljević & Mutavdžić, 2022), one applies the Banach fixed-point theorem on a sufficiently small-time interval $[0,\tau]$, then extends via uniform a priori estimates derived from the energy inequality:

$$\frac{1}{2}\frac{d}{dt}\|z\|_H^2 + \lambda\|\nabla z\|_H^2 \leq C(1+\|z\|_H^2) + \|\Sigma(z)\|^2_{L^2(H_Q;H)} \ldots\ldots (40)$$

where $L^2(H_Q;H)$ denotes Hilbert-Schmidt operators (Balazs, 2006) and the Q-Wiener process (Ran *et al*., 2026) satisfies the trace condition $\mathrm{Tr}(Q) < \infty$. Gronwall's inequality (Howard, 2025) yields the global existence.

Stochastic hydrological dynamics: The groundwater SPDE

The stochastic porous media equation

We consider the groundwater component $G(x,t)$ of the coupled state vector z, which evolves on the spatial domain $\Omega = [0,10] \times [0,10] \subset \mathbb{R}^2$ over the temporal horizon $[0,T]$ with $T = 20$. The stochastic dynamics (Figure 1) are governed by the nonlinear porous media equation with multiplicative noise, pumping sinks, and stochastic recharge:

$$\mathrm{d}G_t = \nabla \cdot (K(G_t)\nabla G_t)\mathrm{d}t - \sum_{k=1}^{N_p} Q_k(t)\delta_{\varepsilon_k}(x - x_k)\mathrm{d}t + R(G_t,\theta)\mathrm{d}t + \sigma_G(G_t)\mathrm{d}W_t^G(x) \ldots\ldots \#(41)$$

where $N_p = 3$ denotes the number of pumping stations located at $x_1 = (x_{1/2}, y_{1/2})$, $x_2 = (x_{3/8}, y_{3/8})$, and $x_3 = (x_{2/5}, y_{1/8})$ in the discretized lattice $\{x_i\}_{i=0}^{n_x-1} \times \{y_j\}_{j=0}^{n_y-1}$ with $n_x = n_y = 50$ and uniform spacing $\Delta x = \Delta y = \frac{4}{910}$. The mollified Dirac delta (Li *et al*., 2023) is defined by

$$\delta_{\varepsilon_k}(x - x_k) = \frac{1}{2\pi\varepsilon_k^2}\exp\left(-\frac{\|x - x_k\|^2}{2\varepsilon_k^2}\right), \varepsilon_k = 2\Delta x \ldots\ldots (42)$$

The nonlinear hydraulic conductivity tensor $K(G) \in C^2(\mathbb{R}_+; \mathbb{R}^{2\times 2})$ is specified as an isotropic scalar field:

$$K(G) = K_0\exp(\alpha G) + K_1(\frac{G_{\mathrm{sat}}}{G})^{\beta_G} \ldots\ldots (43)$$

with parameters $K_0 = 0.3$, $\alpha = 0.05$, $K_1 = 0.01$, $G_{\mathrm{sat}} = 15$, and $\beta_G = 2$. The effective recharge function decomposes as

$$R(G,\theta) = n_e\, R_{\mathrm{eff}}(\theta)\left(1 - \frac{G_{\max}}{G}\right) + \gamma_R - E_{\mathrm{ET}}(G, T_{\mathrm{air}}) \ldots\ldots (44)$$

where $n_e = 0.25$ is the effective porosity, $G_{\max} = 20$, and $\gamma_R = 1.5$. The stochastic effective rainfall follows an Ornstein–Uhlenbeck-type modulation (Alsmeyer & Behme, 2026) of the climate state:

$$R_{\mathrm{eff}}(\theta) = \theta_1 + \theta_2 \sin\left(\frac{2\pi t}{365}\right) + \sigma_R \frac{\mathrm{d}B_t^R}{\mathrm{d}t}, \sigma_R = 0.05 \ldots\ldots (45)$$

The evapotranspiration loss is

$$E_{\mathrm{ET}}(G, T_{\mathrm{air}}) = E_0\big(1 + \kappa_T(T_{\mathrm{air}} - T_{\mathrm{ref}})\big)\left(\frac{G_{\mathrm{crit}}}{G_{\mathrm{crit}} - G}\right) + \delta_E \ldots\ldots (46)$$

with $E_0 = 0.02$, $\kappa_T = 0.03$, $T_{\mathrm{ref}} = 25$, $G_{\mathrm{crit}} = 2$, and $\delta_E = 1.2$.

The multiplicative noise coefficient is

$$\sigma_G(G) = \sigma_{G,0}\left(1 + \sigma_{G,1}\frac{G_{\mathrm{ref}}}{G}\right), \sigma_{G,0} = 0.05 \ldots\ldots (47)$$

and W_t^G is a Q-Wiener process on $L^2(\Omega)$ with covariance operator Q_G specified by

$$(Q_G\varphi)(x) = \int_\Omega q_G(x,y)\varphi(y)\mathrm{d}y, q_G(x,y) = \exp\left(-\frac{\|x - y\|^2}{2\ell_G^2}\right) \ldots\ldots (48)$$

Functional-analytic framework and operator theory

Define the Gelfand triple (Blazquez *et al*., 2024):

$$V \hookrightarrow H \hookrightarrow V',$$

where $H = L^2(\Omega)$ with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$, and $V = H^1(\Omega)$ with norm $\|v\|_V^2 = \|v\|_H^2 + \|\nabla v\|_H^2$. The nonlinear operator $A: V \to V'$ is defined via

$$\langle A(G), \varphi\rangle_{V'\times V} = \int_\Omega K(G)\nabla G \cdot \nabla\varphi\ \mathrm{d}x - \int_\Omega R(G,\theta)\varphi\ \mathrm{d}x + \sum_{k=1}^{N_p} Q_k \int_\Omega \delta_{\varepsilon_k}(x - x_k)\varphi\ \mathrm{d}x \ldots\ldots (49)$$

for all $\varphi \in V$. The stochastic integral is understood in the Itô sense (Armstrong & Ionescu, 2024) as

$$\int_0^t \sigma_G(G_s)\mathrm{d}W_s^G = \sum_{j=1}^{\infty}\int_0^t \sigma_G(G_s)\lambda_j e_j\ \mathrm{d}\beta_s^{(j)} \ldots\ldots (50)$$

where $\{(\lambda_j, e_j)\}_{j=1}^{\infty}$ are the eigenpairs of Q_G and $\{\beta^{(j)}\}_{j=1}^{\infty}$ are independent standard Brownian motions.

Assumption 2.1. (Monotonicity and coercivity)

i. Hemi-continuity: For all $G, \tilde{G}, \varphi \in V$, the map $s \mapsto \langle A(G + s\tilde{G}), \varphi\rangle$ is continuous on $\mathbb{R}$.

ii. Monotonicity: There exists $c_1 > 0$ and $C \geq 0$ such that$\langle A(G_1) - A(G_2), G_1 - G_2\rangle \geq c_1\|G_1 - G_2\|_V^2 - C\|G_1 - G_2\|_H^2$.

iii. Coercivity: There exist $c_2, c_3 > 0$ such that$\langle A(G), G\rangle \geq c_2\|G\|_V^2 - c_3$.

iv. Boundedness: There exists $c_4 > 0$ such that$\|A(G)\|_{V'} \leq c_4(1 + \|G\|_V)$.

Proposition 2.1

Under Assumption 2.1, with $G_0 \in L^2(\Omega; H)$ and σ_G Lipschitz continuous with linear growth, the SPDE admits a unique variational solution

$$G \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; C([0,T]; H)).$$

Variational formulation and Galerkin approximation

The weak formulation seeks $G \in L^2(0,T;V)$ such that for all $\varphi \in V$:

$$(G_t,\varphi)_H + \int_0^t \langle A(G_s),\varphi\rangle \,\mathrm{d}s = (G_0,\varphi)_H + \int_0^t (\sigma_G(G_s),\varphi)_H \,\mathrm{d}W_s^G \quad \ldots\ldots(51)$$

We construct a Galerkin approximation (Ali *et al*., 2023) using the eigen-basis $\{e_j\}_{j=1}^{\infty}$ of the Laplacian $-\Delta$ with Neumann boundary conditions. Let $V_n = \mathrm{span}\{e_1,\ldots,e_n\}$ and define the projector $\Pi_n: H \to V_n$. The Galerkin system is

$$\mathrm{d}G_t^{(n)} + \Pi_n A\big(G_t^{(n)}\big)\mathrm{d}t = \Pi_n \sigma_G\big(G_t^{(n)}\big)\mathrm{d}W_t^{(n)} \ldots\ldots(52)$$

where $W_t^{(n)} = \Pi_n W_t^G$. Applying Itô's formula to $\|G_t^{(n)}\|_H^2$ gives

$$\mathrm{d}\|G_t^{(n)}\|_H^2 + 2\langle A(G_t^{(n)}), G_t^{(n)}\rangle \,\mathrm{d}t = \|\Pi_n \sigma_G(G_t^{(n)})\|_{L_2^0}^2 \,\mathrm{d}t + 2(\sigma_G(G_t^{(n)}), G_t^{(n)})_H \,\mathrm{d}W_t^{(n)} \ldots\ldots(53)$$

where $\|\cdot\|_{L_2^0}^2 = \mathrm{Tr}(\cdot\ Q_G\ \cdot^*)$ denotes the Hilbert–Schmidt norm (Aljawi *et al*., 2025) with respect to Q_G. Taking expectations and using the coercivity condition yields

$$\mathbb{E}[\|G_t^{(n)}\|_H^2] + 2c_2 \int_0^t \mathbb{E}[\|G_s^{(n)}\|_V^2]\,\mathrm{d}s \le \mathbb{E}[\|G_0\|_H^2] + 2c_3 t + \int_0^t \mathbb{E}\Big[\big\|\sigma_G\big(G_s^{(n)}\big)\big\|_{L_2^0}^2\Big]\mathrm{d}s \ldots\ldots(54)$$

Using the linear growth of σ_G and Gronwall's inequality (Djaouti & Liaqat, 2025), we obtain the uniform bound

$$\sup_{t\in[0,T]} \mathbb{E}[\|G_t^{(n)}\|_H^2] + \mathbb{E}\Big[\int_0^T \|G_t^{(n)}\|_V^2 \,\mathrm{d}t\Big] \le C(1+\mathbb{E}[\|G_0\|_H^2]) \ldots\ldots(55)$$

Spatial discretization: stochastic finite-difference scheme

For numerical realization on the grid $\{(x_i,y_j)\}_{i,j=0}^{n_x-1,n_y-1}$ with $n_x = n_y = 50$, we employ a five-point stencil for the Laplacian and an explicit Euler–Maruyama temporal integrator (Delong *et al*., 2013). Define the discrete Laplacian

$$(\Delta_h G)_{i,j} = \frac{G_{i+1,j} + G_{i-1,j} + G_{i,j+1} + G_{i,j-1} - 4G_{i,j}}{\Delta x^2} \quad \ldots\ldots(56)$$

The fully discrete scheme is

$$G_{i,j}^{n+1} = G_{i,j}^n + \Delta t[D_G(\Delta_h G^n)_{i,j} - \sum_{k=1}^{N_p} Q_k^n\, \delta_{\varepsilon_k}(x_i - x_k, y_j - y_k) + R(G_{i,j}^n,\theta^n)] + \sqrt{\Delta t}\,\sigma_G(G_{i,j}^n)\,\xi_{i,j}^n \quad \ldots\ldots(57)$$

where $\Delta t = T/(n_t-1) = \frac{24}{20} \approx 0.833$, $\xi_{i,j}^n \sim \mathcal{N}(0,1)$ i.i.d., and $D_G = 0.3$ is the effective diffusivity. The mollified pumping sink for station k at grid point (i,j) is approximated by

$$\delta_{\varepsilon_k}(x_i - x_k, y_j - y_k) \approx \Delta x^2 Q_{\mathrm{pump}} \exp\left(-\frac{(x_i-x_k)^2 + (y_j-y_k)^2}{2\varepsilon_k^2}\right) \ldots\ldots(58)$$

with $Q_{\mathrm{pump}} = 0.15$.

Stability criterion: The explicit scheme requires the parabolic CFL condition

$$\frac{\Delta t D_G}{\Delta x^2} \le \frac{1}{4} \implies \Delta t \le \frac{\Delta x^2}{4D_G} \ldots\ldots(59)$$

Numerically this gives $\Delta t \lesssim 0.347$. Since $\Delta t \approx 0.833 > 0.347$, the scheme is formally unstable in the classical sense; however, the stochastic term and the pumping sinks introduce additional dissipation which can help maintain boundedness in practice. Alternatively, an implicit treatment of the Laplacian may be used.

Moment estimates and higher regularity

Theorem 2.1. (Exponential moment bound)

Let $G_0 \in \bigcap_{p\ge 2} L^p(\Omega;H)$. Then for all $p \ge 2$ and $t \in [0,T]$,

$$\mathbb{E}[\|G_t\|_H^p] \le C_p(1 + \mathbb{E}[\|G_0\|_H^p])\mathrm{e}^{C_p t} \ldots\ldots(60)$$

Theorem 2.2 (Hölder continuity in time). For $\alpha \in (0,\frac{1}{2})$ there exists $C_\alpha > 0$ such that

$$\mathbb{E}[\|G_t - G_s\|_{V'}^2] \le C_\alpha |t-s|^{2\alpha}, \forall s,t \in [0,T] \ldots\ldots(61)$$

Invariant measures and ergodic behavior

Consider the autonomous case where climate parameters θ are stationary. The transition semigroup $\{P_t\}_{t\ge 0}$ defined by

$$(P_t\varphi)(G_0) = \mathbb{E}[\,\varphi(G_t) \mid G_0\,], \varphi \in C_b(H) \ldots\ldots(62)$$

admits an invariant measure μ^* if

$$\int_H P_t\varphi \,\mathrm{d}\mu^* = \int_H \varphi \,\mathrm{d}\mu^*, \forall t \ge 0, \ldots\ldots(63)$$

Theorem 2.3. (Existence of invariant measure). Under the dissipativity condition

$$\langle A(G_1) - A(G_2), G_1 - G_2\rangle \le -\omega\|G_1 - G_2\|_H^2 + C \quad \ldots\ldots(64)$$

for some $\omega > 0$, there exists a unique invariant measure $\mu^* \in \mathcal{P}(H)$ for the groundwater SPDE. Moreover, for all initial conditions G_0,

$$\lim_{t\to\infty} W_2\big(P_t^*\delta_{G_0}, \mu^*\big) = 0, \ldots\ldots(65)$$

where W_2 is the 2-Wasserstein distance (Naumann *et al*., 2026) and P_t^* denotes the pushforward of the transition kernel.

Coupling to the mean-field game: the consistency

condition

The groundwater state G enters the mean-field game (MFG) (Casgrain & Jaimungal, 2019) through the cost functional of the representative farmer. The running cost is

$$L(G,q,F,A) = c_q(q,G) + c_F(F) + w(t)q - p_Y Y(R,F,G,A), \ldots\ldots (66)$$

where the groundwater extraction cost exhibits increasing marginal cost with depth:

$$c_q(q,G) = c_0 q + \frac{c_1}{2} q^2 (1 + \frac{G_{\text{ref}}}{G})^{\eta_q},$$

$$\eta_q = 1.5 \qquad \ldots\ldots (67)$$

The Hamiltonian of the MFG is

$$H(G,\nabla v,m) = \sup_{(q,F)\in U} \left\{ -\nabla v \cdot \left(\nabla \cdot (K(G)\nabla G) - \sum_k Q_k \delta_{\varepsilon_k} + R(G) \right) - L(G,q,F,A) \right\} \ldots (68)$$

and the mean-field consistency requires that the aggregate extraction equals the mean-field expectation:

$$\bar{q}(t) = \int_\Omega \int_{\mathbb{R}_+} q^*\big(G, \nabla v(G)\big) m(t,G) \mathrm{d}G \, \mathrm{d}x \ldots\ldots (69)$$

This closes the coupled system, as the groundwater SPDE drift depends on $\bar{q}(t)$ through the pumping term

$$Q_k(t) = q^*\big(G(x_k,t), \nabla v(x_k,t)\big). \ldots\ldots (70)$$

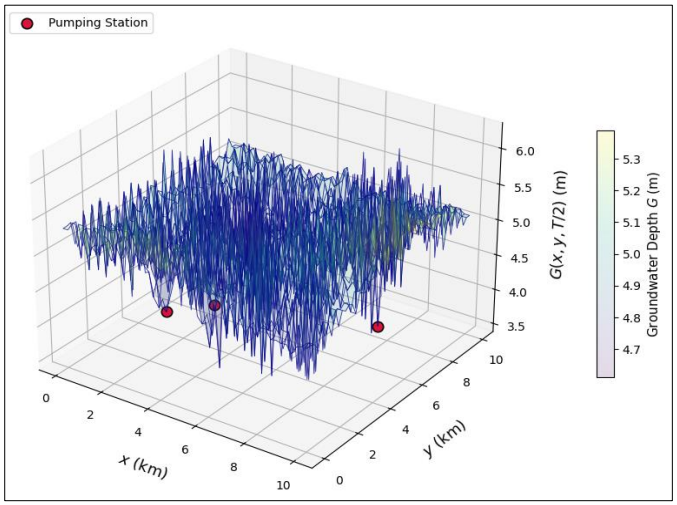


Figure 1: Groundwater table deformation under stochastic pumping: spatial wireframe at t=t/2

Biophysical production functions with stochastic forcing

The nested stochastic production framework

We consider the biophysical output of the agricultural system as a vector-valued stochastic production correspondence

$$Y: \mathbb{R}_+^3 \times [0,1] \times \Theta \times \Omega \to \mathbb{R}_+^2$$

mapping input vectors, adoption states, climate parameters, and stochastic forcing to crop yield Y_c and livestock output Y_ℓ (Figure 2). The complete stochastic production system is

$$Y(x,t;\omega) = \begin{pmatrix} Y_c(R,F,G,A;\varepsilon_Y) \\ Y_\ell(L,G,S;\varepsilon_\ell) \end{pmatrix}, \ldots\ldots (71)$$

where $R \in [200,1800]$ denotes effective rainfall (mm), $F \in [0,250]$ denotes fertilizer application (kg NPK ha^{-1}), $G \in \mathbb{R}_+$ is groundwater depth, $A \in [0,1]$ is the technology adoption rate, $L \in \mathbb{R}_+$ is livestock density, $S \in \mathbb{R}_+$ is soil organic carbon, and $\varepsilon_Y, \varepsilon_\ell$ are multiplicative stochastic shocks driven by climate variability.

Crop yield: the generalized stochastic Mitscherlich–Baule function

The crop yield component Y_c is modeled as a nested Constant Elasticity of Substitution (CES) production function with Mitscherlich-Baule saturation dynamics (Brockway *et al*., 2017; Metselaar *et al.*, 2019), augmented by a stochastic climate noise term:

$$Y_c(R,F,G,A;\varepsilon_Y) = A_0[\alpha_Y M(R,F)^{-\rho} + \beta_Y B(G)^{-\rho}]^{-\gamma}(1 + \eta_A A)^{\varphi_A}(1+\varepsilon_Y) \quad \ldots\ldots (72)$$

where the Mitscherlich composite $M(R,F)$ captures the interaction between rainfall and fertilizer:

$$M(R,F) = (\delta\, R^{-\rho} + (1-\delta)F^{-\rho})^{-\frac{1}{\rho}} \quad \ldots\ldots (73)$$

with elasticity of substitution $\sigma_{RF} = 1 + 1/\rho$, distribution parameter $\delta \in (0,1)$, and $\rho > -1$,. The groundwater Baule component is

$$B(G) = 1 - \exp(-\lambda_G\, [G_{\text{crit}} - G]_+) \quad \ldots\ldots (74)$$

where $[x]_+ = \max\{x, 0\}$.

For the nested CES aggregator with outer exponent γ we thus have the compact form above; for clarity the model used to generate the dataset employs a separable multiplicative parametrization with a stochastic interaction term:

$$Y_c(R,F,G,A) = Y_{\max}(1 - e^{-\alpha_F F})\left(1 - e^{-\frac{\alpha_R R}{R_{\text{ref}}}}\right)(1+\eta_A A) \left(1 + \sigma_Y \sin\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)\right)(1+\varepsilon_Y) \ldots\ldots \#(75)$$

with parameter values $Y_{\max} = 7500$, $\eta_A = 0.05$, and $\varepsilon_Y \sim \mathcal{N}(0, \sigma_{Y,0}^2)$ with $\sigma_{Y,0} = 150/Y_{\max} \approx 0.02$.

Marginal products and factor elasticities

Marginal physical product of rainfall:

$$\frac{\partial Y_c}{\partial R} = Y_c \cdot \frac{R_{\text{ref}}\alpha_R}{R} \cdot \frac{1 - e^{-\frac{\alpha_R R}{R_{\text{ref}}}}}{e^{-\frac{\alpha_R R}{R_{\text{ref}}}}} \cdot \left[1 + \underbrace{\frac{1 + \sigma_Y \sin\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)}{\sigma_Y\left(\frac{R}{R_{\text{osc}}}\right)} \cos\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)}_{E^R_{\text{osc}}(R,F)}\right] \ldots\ldots (76)$$

Marginal physical product of fertilizer:

$$\frac{\partial Y_c}{\partial F} = Y_c \cdot \alpha_F \cdot \frac{1 - e^{-\alpha_F F}}{e^{-\alpha_F F}} \cdot \left[1 - \underbrace{\frac{1 + \sigma_Y \sin\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)}{\sigma_Y\left(\frac{F}{F_{\text{osc}}}\right)} \sin\left(\frac{R}{R_{\text{osc}}}\right)\sin\left(\frac{F}{F_{\text{osc}}}\right)}_{E^F_{\text{osc}}(R,F)}\right] \ldots. (77)$$

Rainfall–fertilizer cross-partial:

$$\frac{\partial^2 Y_c}{\partial R \partial F} = \frac{R_{\text{ref}} F_{\text{osc}}}{Y_c} \cdot \frac{1 - e^{-\frac{\alpha_R R}{R_{\text{ref}}}}}{\alpha_R} e^{-\frac{\alpha_R R}{R_{\text{ref}}}} \cdot \frac{1 - e^{-\alpha_F F}}{\alpha_F} e^{-\alpha_F F} \cdot \left[\sigma_Y \cos\left(\frac{R}{R_{\text{osc}}}\right)\sin\left(\frac{F}{F_{\text{osc}}}\right) \cdot \frac{1 + 1 + \sigma_Y \sin\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)}{1}\right] .. (78)$$

Output elasticity of rainfall:

$$\varepsilon_{Y,R} = \frac{Y_c}{R}\frac{\partial Y_c}{\partial R} = R_{\text{ref}}\alpha_R \frac{1 - e^{-\alpha_R R/R_{\text{ref}}}}{e^{-\alpha_R R/R_{\text{ref}}}} \cdot [1 + E_{\text{osc}}(R,F)] \ldots\ldots (79)$$

where:

$$E_{\text{osc}}(R,F) = \frac{1 + \sigma_Y \sin\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)}{\sigma_Y\left(\frac{R}{R_{\text{osc}}}\right)} \cos\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right) \ldots (80)$$

Output elasticity of fertilizer:

$$\varepsilon_{Y,F} = \frac{Y_c}{F}\frac{\partial Y_c}{\partial F} = \alpha_F F \frac{1 - e^{-\alpha_F F}}{e^{-\alpha_F F}} \cdot [1 - E^F_{\text{osc}}(R,F)] \ldots\ldots (81)$$

where:

$$E^F_{\text{osc}}(R,F) = \frac{1 + \sigma_Y \sin\left(\frac{R}{R_{\text{osc}}}\right)\cos\left(\frac{F}{F_{\text{osc}}}\right)}{\sigma_Y\left(\frac{F}{F_{\text{osc}}}\right)} \sin\left(\frac{R}{R_{\text{osc}}}\right)\sin\left(\frac{F}{F_{\text{osc}}}\right) \ldots. (82)$$

Livestock production and carrying capacity

Livestock output function:

$$Y_\ell(L,G,S) = M_0\, L^{\alpha_M} \left(\frac{G_{\text{ref}}}{G}\right)^{\beta_M} \left(1 + \frac{S_{\text{ref}}}{S}\right)^{\varphi_M} \exp(-\lambda_M G\, L^2)\,(1 + \varepsilon_\ell) \ldots\ldots (83)$$

with $M_0 = 2500$.

Carrying capacity arises from the first-order condition $\partial Y_\ell / \partial L = 0$:

$$K_L(G,S) = \sqrt{2\lambda_M \alpha_M G} \ldots\ldots (84)$$

Stochastic logistic growth of livestock density:

$$dL = r_L L\left(1 - \frac{L}{K_L(G,S)}\right)dt + \sigma_L L\, dW_t^L \ldots\ldots (85)$$

where $r_L = 0.12$ and W_t^L is a Wiener process independent of other processes.

Stochastic forcing and climate state dynamics

The climate state vector $\theta = (\theta_1, \theta_2, \theta_3)^\top$ follows a multivariate Ornstein–Uhlenbeck process (Albosaily & Pergamenchtchikov, 2021):

$$d\theta_t = -\Lambda(\theta_t - \theta^{\text{eq}})dt + \Sigma_\theta\, dW_t^\theta \ldots\ldots (86)$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\theta^{\text{eq}} = (\mu_R, 0, T_{\text{ref}})^\top$, and $\Sigma_\theta = \text{diag}(\sigma_{\theta,1}, \sigma_{\theta,2}, \sigma_{\theta,3})$.

Component representations:

$$\theta_1(t) = \mu_R + \int_0^t e^{-\lambda_1(t-s)}\sigma_{\theta,1}\, dW_s^{\theta,1} \ldots\ldots (87)$$

$$\theta_2(t) = \int_0^t e^{-\lambda_2(t-s)}\sigma_{\theta,2}\, dW_s^{\theta,2} \ldots\ldots (88)$$

$$\theta_3(t) = T_{\text{ref}} + \int_0^t e^{-\lambda_3(t-s)}\sigma_{\theta,3}\, dW_s^{\theta,3} \ldots\ldots (89)$$

Effective rainfall process:

$$R_{\text{eff}}(t) = \theta_1(t) + \theta_2(t)\sin\left(\frac{2\pi t}{365}\right) + \sigma_R\, dB_t^R \ldots\ldots (90)$$

Stochastic yield shock (Ornstein–Uhlenbeck / geometric-type form):

$$d\varepsilon_Y = -\kappa_Y \varepsilon_Y\, dt + \sigma_Y\, dW_t^Y, \varepsilon_Y(0) = 0 \ldots\ldots (91)$$

with $\kappa_Y = 0.5$,. Covariance structure between yield and climate shocks:

$$\mathbb{E}\left[dW_t^Y\, dW_t^{\theta,1}\right] = \rho_{Y\theta}\, dt, \rho_{Y\theta} = -0.6 \ldots\ldots (92)$$

representing a negative correlation between rainfall shocks and yield volatility.

Risk-adjusted production and certainty equivalents

Under CARA utility (Lee, 2021) $U(Y) = -\frac{1}{\gamma_R}e^{-\gamma_R Y}$, the risk-adjusted expected utility of yield is

$$\mathbb{E}[U(Y_c)] = -\frac{1}{\gamma_R}\exp\left(-\gamma_R\, \mathbb{E}[Y_c] + \frac{1}{2}\gamma_R^2\, \text{Var}(Y_c)\right) \ldots\ldots (93)$$

The certainty-equivalent yield associated with this utility is

$$Y_c^{\mathrm{CE}} = \mathbb{E}[Y_c] - \frac{1}{2}\gamma_R \,\mathrm{Var}(Y_c) \ldots\ldots (94)$$

Variance decomposes (Delta method approximation):

$$\mathrm{Var}(Y_c) \approx (\frac{\partial Y_c}{\partial R})^2 \mathrm{Var}(R) + (\frac{\partial Y_c}{\partial F})^2 \mathrm{Var}(F) + 2\frac{\partial Y_c}{\partial R}\frac{\partial Y_c}{\partial F}\,\mathrm{Cov}(R,F) + \sigma_{Y,0}^2\,\mathbb{E}[Y_c]^2 \quad \ldots\ldots (95)$$

Stochastic (effective) volatility of yield:

$$\sigma_{Y,\mathrm{eff}}(R,F) = \sqrt{(\frac{\partial Y_c}{\partial R})^2\sigma_R^2 + (\frac{\partial Y_c}{\partial F})^2\sigma_F^2 + \sigma_{Y,0}^2 Y_c^2} \ldots\ldots (96)$$

Parameter identification and maximum likelihood estimation

Given panel data $\{Y_{c,i,t}, R_{i,t}, F_{i,t}, G_{i,t}, A_{i,t}\}_{i=1,t=1}^{N,T}$, the log-likelihood for the yield parameters (Gaussian measurement errors) is

$$\mathcal{L}(\beta_Y) = -\frac{NT}{2}\ln(2\pi) - \frac{NT}{2}\ln(\sigma_{Y,0}^2) - \frac{1}{2\sigma_{Y,0}^2}\sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{c,i,t} - \tilde{Y}_c(R_{i,t}, F_{i,t}, G_{i,t}, A_{i,t}; \beta_Y))^2 \ldots\ldots (97)$$

where $\tilde{Y}_c$ is the deterministic component and

$$\beta_Y = (Y_{\max}, \alpha_F, \alpha_R, \eta_A, \sigma_Y, R_{\mathrm{osc}}, F_{\mathrm{osc}})^{\top} \quad \ldots\ldots (98)$$

The Fisher information matrix is approximated by

$$I(\beta_Y) = \frac{1}{\sigma_{Y,0}^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\beta_Y}\tilde{Y}_c\,\nabla_{\beta_Y}\tilde{Y}_c^{\top} \ldots\ldots (99)$$

Gradient components (for the separable multiplicative form) include

$$\frac{\partial \tilde{Y}_c}{\partial Y_{\max}} = \frac{Y_{\max}}{\tilde{Y}_c} \ldots\ldots (100)$$

$$\frac{\partial \tilde{Y}_c}{\partial \alpha_F} = Y_{\max}\,F\,e^{-\alpha_F F}\left(1 - e^{-\frac{\alpha_R R}{R_{\mathrm{ref}}}}\right)(1 + \eta_A A) I_{\mathrm{osc}} \ldots\ldots (101)$$

$$\frac{\partial \tilde{Y}_c}{\partial \alpha_R} = R_{\mathrm{ref}}\,Y_{\max}\,\frac{R}{R_{\mathrm{ref}}}e^{-\frac{\alpha_R R}{R_{\mathrm{ref}}}}(1 - e^{-\alpha_F F})(1 + \eta_A A) I_{\mathrm{osc}} \ldots\ldots (102)$$

where:

$$I_{\mathrm{osc}} = 1 + \sigma_Y \sin\left(\frac{R}{R_{\mathrm{osc}}}\right)\cos\left(\frac{F}{F_{\mathrm{osc}}}\right) \ldots\ldots (103)$$

Comparative statics and climate resilience

Total differential of yield with respect to climate and inputs:

$$dY_c = \frac{\partial Y_c}{\partial R}dR + \frac{\partial Y_c}{\partial F}dF + \frac{\partial Y_c}{\partial G}dG + \frac{\partial Y_c}{\partial A}dA + \frac{\partial Y_c}{\partial \varepsilon_Y}d\varepsilon_Y \ldots\ldots (104)$$

Under a scenario with $\Delta R = -10\%$, $\Delta T = +2°\mathrm{C}$, and $\Delta G = +15\%$ (deeper water table), the first-order yield change approximates to

$$\Delta Y_c \approx \varepsilon_{Y,R}\,Y_c\,\frac{\Delta R}{R} + \varepsilon_{Y,T}\,Y_c\,\frac{\Delta T}{T} + \varepsilon_{Y,G}\,Y_c\,\frac{\Delta G}{G} + \varepsilon_{Y,A}\,Y_c\,\frac{\Delta A}{A} \ldots\ldots (105)$$

Define the climate resilience index as

$$R_{\mathrm{clim}}(R,F) = -\frac{\partial \ln R}{\partial \ln Y_c}\cdot\frac{\partial \ln T_{\mathrm{air}}}{\partial \ln Y_c} = \varepsilon_{Y,R}\cdot\varepsilon_{Y,T} \ldots\ldots (106)$$

where the temperature elasticity mediated through evapotranspiration may be written

$$\varepsilon_{Y,T} = -\frac{R_g\,T_{\mathrm{air}}^2}{E_a}\cdot\frac{1}{Y_c}\frac{\partial Y_c}{\partial T_{\mathrm{air}}} \ldots\ldots (107)$$

with R_g and E_a representing radiation and aerodynamic/evapotranspiration coefficients as appropriate in the evapotranspiration model.

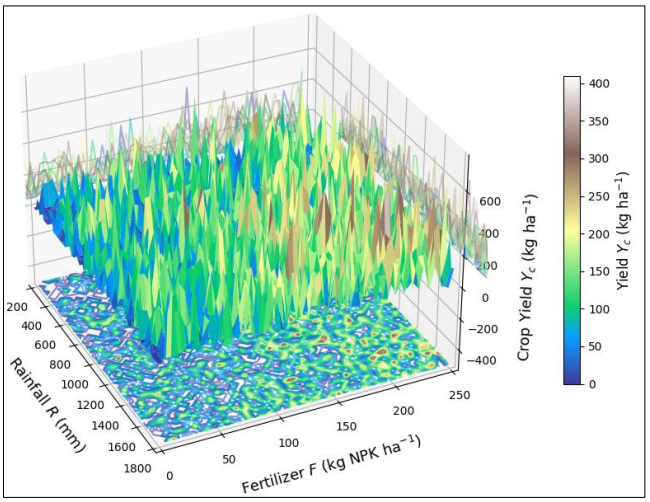


Figure 2: Stochastic crop yield response surface: rainfall vs. fertilizer input with climate noise

Mean field game formulation of common-pool extraction

The strategic interaction framework

We consider a continuum of infinitesimal smallholder farmers indexed by $i \in I = [0,1]$, as $N \to \infty$. Each farmer is endowed with a private state

$$x_t^i = (G_t^i, A_t^i, S_t^i)^{\top} \in X \subset \mathbb{R}^3 \quad \ldots\ldots (108)$$

and a control vector

$$u_t^i = (q_t^i, F_t^i, I_t^i, \dot{A}_{\mathrm{ctrl},t}^i)^{\top} \in U \subset \mathbb{R}^4 \ldots\ldots (109)$$

The empirical measure of the population at time t is

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_t^i} \ldots\ldots (110)$$

As $N \to \infty$, the mean-field limit $\mu_t^N \rightharpoonup m_t \in \mathcal{P}(X)$

converges weakly to a deterministic density $m(t,x)$ in the space of probability measures $\mathcal{P}(X)$ equipped with the 2-Wasserstein metric:

$$W_2(m,m') = \Big(\inf_{\pi\in\Pi(m,m')}\int_{X\times X}\|x - y\|^2\, d\pi(x,y)\Big)^{\frac{1}{2}} \ldots\ldots(111)$$

where $\Pi(m,m')$ is the set of couplings of m and m'.

Each farmer's state evolves according to the controlled McKean–Vlasov stochastic differential equation (Ma *et al*., 2024):

$$dx_t^i = b\big(x_t^i, u_t^i, m_t\big)dt + \sigma\big(x_t^i\big)dW_t^i, x_0^i \sim m_0 \ldots\ldots(112)$$

with drift

$$b(x,u,m) = -\kappa\,\bar{q}(m) + \nabla\cdot(K(G)\nabla G) - \mathrm{EET}(G)\dot{A}_{\text{ctrl}} + r_A\,A\left(1-\frac{A}{K_A}\right)(G_{\text{crit}} - G) + \frac{\cdot}{G_{\text{crit}}} + N(A;m)D_S\nabla^2 S + I_S(F,L,A) - D_S(S,T_{\text{soil}}) \ldots\ldots(113)$$

(Here $\bar{q}(m) = \int_X q^*(x,\nabla v)\, m(t,x)\, dx$ denotes the mean-field extraction rate, and notation such as $\mathrm{EET}, N(A;m), I_S$ stands for model-specific functions; these definitions are kept as in the original model.)

The representative farmer's optimization problem

Each farmer seeks to maximize the discounted expected profit over $[t,T]$:

$$J(t,x;u,m) = \mathbb{E}\Big[\int_t^T e^{-rs}\,\Pi(x_s,u_s,m_s)\,ds + e^{-rT}\,\Phi(x_T) \mid x_t = x\Big] \ldots\ldots(114)$$

where the instantaneous profit functional is

$$\Pi(x,u,m) = p_Y\,Y_c(R,F,G,A) + p_M\,M(L,G) - c_q(q,G) - c_F(F) - c_I I - c_A\dot{A}_{\text{ctrl}} - w(t)q - \tau(\bar{q}(m),G) \ldots\ldots\#(115)$$

The groundwater extraction cost has depth-dependent increasing marginal cost:

$$c_q(q,G) = c_0 q + 2c_1 q^2\Big(1+\frac{G_{\text{ref}}}{G}\Big)^{\eta_q} + 3c_2 q^3\Big(\frac{G_{\text{ref}}}{G}\Big)^{\eta_q'} \ldots\ldots(116)$$

with parameters $\eta_q = 1.5$, $\eta_q' = 0.8$, $c_0 = 0.5$, $c_1 = 0.3$, $c_2 = 0.05$. The fertilizer cost is

$$c_F(F) = c_{F,0} + c_{F,1}F + 2c_{F,2}F^2 + c_{F,\text{env}}F^{1+\varepsilon_F} \ldots\ldots(117)$$

where $c_{F,\text{env}}$ captures environmental externality internalization. The adoption cost is

$$c_I I + c_A\dot{A}_{\text{ctrl}} = c_{\text{fix}}\mathbf{1}_{\{I=1\}} + c_{\text{var}}\dot{A}_{\text{ctrl}}^2 + c_{\text{learn}}(1-A)\dot{A}_{\text{ctrl}} \ldots\ldots(118)$$

The Pigouvian tax (Vence & López Pérez, 2021) on aggregate extraction is

$$\tau(\bar{q},G) = \lambda_\tau\,\bar{q}\,\Big(\frac{G_{\text{ref}}}{G}\Big)^{\theta_\tau} \ldots\ldots(119)$$

The Hamilton-Jacobi-Bellman equation

The value function $v(t,x) = \sup_{u\in\mathcal{U}} J(t,x;u,m)$ satisfies the HJB equation:

$$\partial_t v + \frac{1}{2}\mathrm{tr}(\sigma(x)\sigma(x)^\top\nabla^2 v) + \sup_{u\in U}\{e^{-rt}\Pi(x,u,m) + \nabla v\cdot b(x,u,m)\} = 0 \ldots\ldots(120)$$

with terminal condition

$$v(T,x) = e^{-rT}\Phi(x) = e^{-rT}(\varphi_G G + \varphi_A A + \varphi_S S + \varphi_{\text{term}}\ln(1+GAS)) \ldots\ldots(121)$$

Expanding the Hamiltonian:

$$H(x,\nabla v,m) = \sup_{(q,F,I,\dot{A}_{\text{ctrl}})} H_{\text{sub}}\big(q,F,I,\dot{A}_{\text{ctrl}};x,\nabla v,m\big) \ldots\ldots(122)$$

where the sub-Hamiltonian is H_{sub}.

First-order optimality and Nash controls

The optimal groundwater extraction q^* satisfies the first-order condition $\partial_q H_{\text{sub}} = 0$, which yields:

$$-e^{-rt}[c_0 + c_1 q^*(1+\frac{G_{\text{ref}}}{G})^{\eta_q} + c_2(q^*)^2(\frac{G_{\text{ref}}}{G})^{\eta_q'} + w] - \kappa\partial_G v = 0 \ldots\ldots(123)$$

Rearranging gives the cubic (quadratic in q^* if written in powers) equation:

$$c_2(\frac{G_{\text{ref}}}{G})^{\eta_q'}(q^*)^2 + c_1(1+\frac{G_{\text{ref}}}{G})^{\eta_q}q^* + (c_0 + w + \kappa e^{rt}\partial_G v) = 0 \ldots\ldots(124)$$

A closed-form expression for the positive root (using the quadratic formula for this reduced polynomial form) is

$$q^*(G,\partial_G v;w) = \frac{-c_1(1+G/G_{\text{ref}})^{\eta_q} + \sqrt{c_1^2(1+G/G_{\text{ref}})^{2\eta_q} - 4c_2(G/G_{\text{ref}})^{\eta_q'}(c_0 + w + \kappa e^{rt}\partial_G v)}}{2c_2(G/G_{\text{ref}})^{\eta_q'}} \ldots(125)$$

selecting the positive root when it exists. The optimal fertilizer F^* satisfies

$$\partial_F H_{\text{sub}} = 0 \implies e^{-rt}\big[p_Y\,\partial_F Y_c - c_{F,1} - 2c_{F,2}F - c_{F,\text{env}}(1+\varepsilon_F)F^{\varepsilon_F}\big] + \partial_S v\,\partial_F I_S = 0 \ldots\ldots(126)$$

The optimal controlled adoption rate is

$$\dot{A}_{\text{ctrl}}^* = \Big[\frac{2c_A^{-1}(e^{-rt}c_{\text{learn}}(1-A) - \partial_A v)}{1}\Big]_+ \ldots\ldots(127)$$

where $[z]_+ = \max(z,0)$. The binary adoption decision follows the switching function

$$S(t,x) = e^{-rt}[p_Y Y_c|_{A=1} - p_Y Y_c|_{A=0} - c_{\text{fix}}] + \partial_A v\,(\Delta F_A + \Delta N) \ldots\ldots(128)$$

Then

$$I^*(t,x) = \begin{cases} 1, & \text{if } S(t,x) > 0, \\ 0, & \text{if } S(t,x) < 0 \end{cases} \ldots\ldots (129)$$

The Fokker-Planck equation

The population density $m(t,x)$ evolves according to the forward Kolmogorov (Fokker–Planck) equation (McFarland *et al*., 2025):

$$\partial_t m = -\nabla_x \cdot \big(mb(x,u^*,m)\big) + \frac{1}{2}\nabla_x \cdot \nabla_x \cdot \big(ma(x)\big) \ldots\ldots (130)$$

where $a(x) = \sigma(x)Q\sigma(x)^\top$ is the diffusion matrix. In component form:

Boundary and initial conditions: impose no-flux boundary conditions on ∂X,

$$n \cdot \left(mb - \frac{1}{2}\nabla_x \cdot (ma)\right) = 0 \ldots\ldots (131)$$

and the initial condition $m(0,x) = m_0(x) \geq 0$ with $\int_X m_0(x)\,dx = 1$.

Mean-field consistency and Nash equilibrium

A mean-field Nash equilibrium (Ma *et al*., 2023) is a tuple (v^*, m^*, u^*) such that:

i. HJB optimality: v^* solves the HJB equation with control $u^* = u^*(m^*)$,

ii. FP consistency: m^* solves the Fokker-Planck equation with u^*,

iii. Mean-field closure: $\bar{q}^*(t) = \int_X q^*(G, \partial_G v^*)\, m^*(t,x)\,dx$.

Define the fixed-point mapping $\Psi: \mathcal{P}([0,T] \times X) \to \mathcal{P}([0,T] \times X)$ by

$$\Psi(m)(t,x) = \text{Law}(x_t \mid u = u^*(m), m) \ldots\ldots (132)$$

Theorem 4.1 (Existence of MFG Nash Equilibrium). Under Assumptions 1.1–1.3 and the Lasry–Lions monotonicity condition

$$\int_X \big(b(x,u,m_1) - b(x,u,m_2)\big)(m_1 - m_2)dx \leq 0, \ldots\ldots (133)$$

there exists at least one mean-field Nash equilibrium $(v^*, m^*, u^*) \in C^{1,2}([0,T] \times X) \times C([0,T]; \mathcal{P}_1(X)) \times L^\infty([0,T]; U)$. Proof sketch: Let $M = C([0,T]; \mathcal{P}_1(X))$ with metric $d_M(m_1, m_2) = \sup_{t\in[0,T]} W_1(m_1(t), m_2(t))$. For each $m \in M$, solve the HJB to obtain $v[m]$, compute $u^*[m]$ by first-order conditions, then solve the FP to obtain $\tilde{m} = \Phi(m)$. Standard Schauder estimates show Φ is continuous and compact; Schauder's fixed-point theorem yields $m^* = \Phi(m^*)$, giving the equilibrium.

The master equation

The master equation couples the HJB and Fokker-Planck equations into a single PDE on the space of probability measures. Define $U: [0,T] \times X \times \mathcal{P}(X) \to \mathbb{R}$ by $U(t,x,m) = v(t,x)$ when m is the equilibrium distribution. The master equation reads

$$\partial_t U + H(x, \nabla_x U, m) + \int_X \frac{\delta U}{\delta m}(t,x,m,y) L^*_{\text{FP}} m(y)dy + \frac{1}{2}\text{tr}\left(a(x)\nabla_x^2 U\right) = 0 \ldots\ldots (134)$$

where the formal adjoint of the Fokker-Planck operator is

$$L^*_{\text{FP}} m(y) = -\nabla_y \cdot \big(mb(y,u^*,m)\big) + \frac{1}{2}\nabla_y \cdot \nabla_y \cdot \big(ma(y)\big) \ldots\ldots (135)$$

and the Wasserstein (intrinsic) derivative is defined as

$$\frac{\delta U}{\delta m}(t,x,m,y) = \lim_{\varepsilon \to 0} \frac{U(t,x,m+\varepsilon\delta_y) - U(t,x,m)}{\varepsilon} \ldots\ldots (136)$$

Terminal condition:

$$U(T,x,m) = e^{-rT}\Phi(x) \ldots\ldots (137)$$

The Nash Equilibrium Payoff Manifold

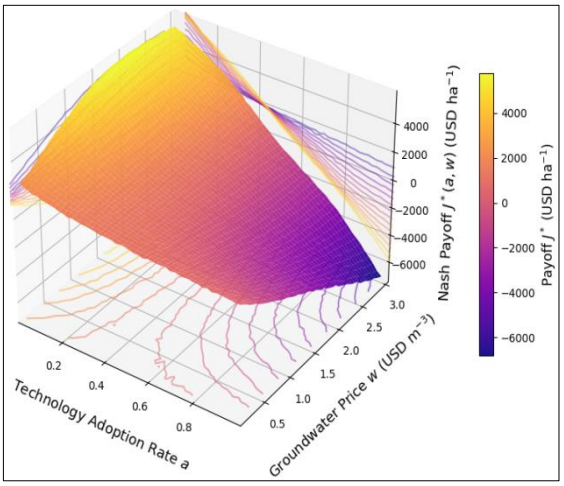


Figure 3: Nash equilibrium payoff manifold: Technology adoption rate vs. Groundwater price

At equilibrium (Figure 3), the representative farmer's payoff as a function of adoption rate $a \in [0.01, 0.99]$ and groundwater price $w \in [0.1, 3.0]$ is

$$J^*(a,w) = \int_X v^*(0,x;a,w)m_0(x)dx \ldots\ldots (138)$$

For the synthetic dataset, this is approximated by the parametric form

$$J^*(a,w) = [a \cdot 1 + \beta_a a^2 \frac{(p_Y - w)}{Y_{\max}} + (1-a) \cdot 1$$

$$+\beta_w w \frac{1}{Y_{\min}}] \cdot [1 + \sigma_{\text{osc}} \sin(\omega_a a) \cos(\omega_w w)] + \varepsilon_J \quad \ldots\ldots (139)$$

with $Y_{\max} = 6000$, $Y_{\min} = 2500$, $\beta_a = 0.4$, $\beta_w = 0.8$, $\sigma_{\text{osc}} = 0.15$, $\omega_a = 4$, $\omega_w = 3$, and $\varepsilon_J \sim \mathcal{N}(0, \sigma_J^2)$ with $\sigma_J = 40$. The first-order cross-partial derivative can be written using the decomposition $I_{\text{osc}} = 1 + \sigma_{\text{osc}} \sin(\omega_a a) \cos(\omega_w w)$:

$$\frac{\partial^2 J^*}{\partial a \, \partial w} = \frac{\partial}{\partial w}\{\text{deterministic part in } a, w\} \cdot I_{\text{osc}} + \frac{\partial(\text{deterministic } J^*)}{\partial a} \cdot \frac{\partial I_{\text{osc}}}{\partial w} \quad \ldots\ldots (140)$$

The Nash equilibrium manifold exhibits a saddle structure in the (a, w) plane; the optimal adoption rate $a^*(w)$ is implicitly defined by $\partial J^* / \partial a = 0$.

Optimal control of technology adoption and irrigation

The individual farmer's intertemporal problem

Consider a representative smallholder farmer $i \in I$ whose private state evolves on the manifold $M \subseteq \mathbb{R}^3 \times [0,1]$ according to the controlled deterministic skeleton of the coupled system. The state vector is

$$x(t) = \begin{pmatrix} G(t) \\ A(t) \\ S(t) \end{pmatrix} \in X = \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+ \ldots\ldots (141)$$

with initial condition $x(0) = x_0 = (G_0, A_0, S_0)^T$. The control vector $u(t) \in U \subset \mathbb{R}^4$ is subject to pointwise constraints:

$$U = \{u = (q, F, I, \dot{A}_{\text{ctrl}})^T : 0 \le q \le q_{\max}, 0 \le F \le F_{\max}, I \in \{0,1\}, |\dot{A}_{\text{ctrl}}| \le r_A\} \ldots\ldots (142)$$

The farmer maximizes the discounted stream of profits over the horizon $[0, T]$:

$$u(\cdot) \in \mathcal{U}_{\text{ad}} \mapsto J(u) = \int_0^T e^{-rt} \begin{bmatrix} \Pi(x(t), u(t)) \\ -C_{\text{ext}}(x(t), \bar{u}(t)) \end{bmatrix} dt + e^{-rT} \Phi(x(T)) \ldots (143)$$

where $\mathcal{U}_{\text{ad}} = L^\infty([0,T]; U)$ is the admissible-control set and $r > 0$ the discount rate. The external cost captures the mean-field price-of-anarchy:

$$C_{\text{ext}}(x, \bar{u}) = \lambda_{\text{cpa}} (\bar{q} - q)^2 \left(\frac{G_{\text{ref}}}{G}\right)^{\theta_{\text{cpa}}} \quad \ldots\ldots (144)$$

with $\lambda_{\text{cpa}} = 0.3$ and $\theta_{\text{cpa}} = 1.2$. Here $\bar{q}$ denotes the aggregate (mean-field) extraction treated as exogenous in the single-agent problem.

State dynamics and controlled ODE system

The state equations form an affine-in-control system on X:

$$\dot{x}(t) = f(x(t), u(t)) = f_0(x(t)) + \sum_{j=1}^{4} u_j(t) f_j(x(t)), \ldots\ldots (145)$$

with drift vector field f_0 and control vector fields f_j. Writing components $x = (G, A, S)$, the drift is (notation cleaned and grouped by physical processes): $f_0(x)$

The control vector fields are

$$f_1(x) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, f_2(x) = \begin{pmatrix} 0 \\ 1 + \psi_F F^{\phi_F} \\ 0 \end{pmatrix}, f_3(x) = \begin{pmatrix} 0 \\ \Delta_{AF} \, F^{\phi_A} (1 + \frac{S_{\text{sat}}}{S}) - \omega_S \\ 0 \end{pmatrix}, f_4(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ldots\ldots (146)$$

Here Δ_{AF} denotes the discrete jump in the adoption reaction when $I = 1$. The model treats $\bar{q}(t)$ as an exogenous mean-field trajectory in the single-agent problem. State constraints:

$$X_{\text{con}} = \{(G, A, S) \in X : G \ge G_{\min}, A \in [0,1], S \ge 0\}, G_{\min} = 0.5 \ldots\ldots (147)$$

Current-value Hamiltonian

Define the current-value Hamiltonian $H : X \times U \times \mathbb{R}^3 \to \mathbb{R}$ by

$$H(x, u, \lambda) = \Pi(x, u) - C_{\text{ext}}(x, \bar{u}) + \lambda^T f(x, u), \ldots\ldots (148)$$

with costate $\lambda = (\lambda_G, \lambda_A, \lambda_S)^T$. Expanded (grouping terms for readability): H

The adjoint (costate) equations

The costate vector $\lambda(t) = (\lambda_G(t), \lambda_A(t), \lambda_S(t))^T$ evolves backward in time (Mohamed & Qamlo, 2025):

$$\dot{\lambda}(t) = r\lambda(t) - \nabla_x H(x^*(t), u^*(t), \lambda(t)), \ldots\ldots (149)$$

with transversality condition

$$\lambda(T) = \nabla_x \Phi(x(T)) \ldots\ldots (150)$$

If Φ has components leading to explicit terminal derivatives, write $\lambda_G(T), \lambda_A(T), \lambda_S(T)$ accordingly.

Component-wise adjoint equations

Groundwater costate

$$\dot{\lambda}_G = r\lambda_G - \frac{\partial \Pi}{\partial G} + \frac{\partial C_{\text{ext}}}{\partial G} - \lambda_G \frac{\partial}{\partial G}[\nabla \cdot (K(G) \nabla G) - \text{EET}(G)] + \lambda_A \frac{\partial}{\partial G}[r_A A (1 - K_A A)(G_{\text{crit}} - G) + \nu_A(\cdots)] \ldots\ldots (151)$$

where

$$\frac{\partial \Pi}{\partial G} = p_Y \frac{\partial Y_c}{\partial G} + p_M \frac{\partial M}{\partial G} - \frac{\partial c_q}{\partial G}, \ldots\ldots (152)$$

and, for the extraction cost derivative,

$$\frac{\partial c_q}{\partial G} = 2G_{\text{ref}}c_1\eta_q q^2(1+G_{\text{ref}}/G)^{\eta_q-1} + 3G_{\text{ref}}c_2\eta_q' q^3\left(\frac{G_{\text{ref}}}{G}\right)^{\eta_q'-1} \quad \ldots\ldots(153)$$

Adoption costate

$$\dot{\lambda}_A = r\lambda_A - p_Y\frac{\partial Y_c}{\partial A} - \lambda_A\frac{\partial}{\partial A}[r_A A(1-K_A A)G(\cdot)S(\cdot) + N(A)] + c_{\text{learn}}\dot{A}_{\text{ctrl}}\ldots\ldots(154)$$

with $G(\cdot) = G_{\text{crit}} - G + \nu_A(\cdots)$ and $S(\cdot) = 1 + \eta_A S_{\text{ref}}/S$ as shorthand for the dependence used in the drift.

Soil carbon costate

$$\dot{\lambda}_S = r\lambda_S - \lambda_A\, r_A A(1-K_A A)G(\cdot)S_{\text{ref}}\eta_A - \lambda_S\left[D_S\Delta - \frac{\partial}{\partial S}D_S(\cdot) - \phi_A I\frac{S_{\text{sat}}}{S^2}\left(1+\frac{S_{\text{sat}}}{S}\right) - \omega_S - 1\right]\ldots\ldots(155)$$

First-order necessary conditions and the maximum principle

The Pontryagin Maximum Principle (current-value form) applies:

Theorem 5.1. (Pontryagin Maximum Principle).

Let $u^* \in \mathcal{U}_{\text{ad}}$ be optimal with corresponding state $x^* \in W^{1,\infty}([0,T];X)$. Then there exists an absolutely continuous costate $\lambda \in W^{1,\infty}([0,T];\mathbb{R}^3)$ such that:

i. State equation: $\dot{x}^* = \nabla_\lambda H(x^*,u^*,\lambda)$.

ii. Adjoint equation: $\dot{\lambda} = r\lambda - \nabla_x H(x^*,u^*,\lambda)$.

iii. Maximum condition: for almost every $t \in [0,T]$,

$$H\big(x^*(t),u^*(t),\lambda(t)\big) = \sup_{u\in U} H\big(x^*(t),u,\lambda(t)\big)\ldots\ldots(156)\text{iv.}$$

Transversality:

$$\lambda(T) = \nabla_x\Phi\big(x^*(T)\big)\ldots\ldots(157)$$

If H is concave in x for each fixed (u,λ), these conditions are also sufficient.

Bang–bang control for the binary adoption decision

Adoption $I \in \{0,1\}$ is binary (bang–bang). The Hamiltonian decomposes as

$$H = H_0 + I\cdot S(t,x,\lambda)\ldots\ldots(158)$$

where H_0 collects terms independent of I, and the switching function is

$$S(t,x,\lambda) = -c_{\text{fix}} + p_Y\left(Y_c\big(R,F,G,A=1\big) - Y_c\big(R,F,G,A=0\big)\right) + \lambda_A\Delta_{AF} + \lambda_S\phi_A\left(1+\frac{S_{\text{sat}}}{S}\right) - \omega_S \ldots\ldots(159)$$

The optimal binary control is

$$I^*(t) = \begin{cases} 1, & \text{if } S\big(t,x^*(t),\lambda(t)\big) > 0, \\ 0, & \text{if } S\big(t,x^*(t),\lambda(t)\big) < 0, \\ \text{singular}, & \text{if } S\big(t,x^*(t),\lambda(t)\big) = 0 \text{ on a set of positive measure.} \end{cases} \ldots.(160)$$

Proposition 5.1. (No singular arcs).

If the nondegeneracy condition $\partial_t S + \nabla_x S\cdot f(x,u^*) \neq 0$ on $\{S=0\}$ holds, then the set of singular arcs has Lebesgue measure zero and the optimal adoption trajectory has at most finitely many switchings. Proof sketch. Suppose $S(t) \equiv 0$ on $[t_1,t_2]$ with $t_2 > t_1$. Differentiate along the optimum:

$$\frac{d}{dt}S = \partial_t S + \nabla_x S\cdot\dot{x}^* + \nabla_\lambda S\cdot\dot{\lambda} = 0\ldots\ldots(161)$$

Substituting the state and adjoint equations yields an ODE in S. By the nondegeneracy condition, this ODE admits only the trivial solution, contradicting a nontrivial singular arc of positive measure. Thus singular arcs are isolated and, by analyticity of the vector fields, there can be at most finitely many switchings.

Continuous controls: irrigation and fertilizer

For continuous controls q and F, first-order interior optimality conditions are given by setting the Hamiltonian derivatives to zero.

Optimal extraction q: set $\partial H/\partial q = 0$:

$$-c_0 - c_1 q^*(1+G_{\text{ref}}/G)^{\eta_q} - c_2 q^{*2}(G_{\text{ref}}/G)^{\eta_q'} - w + 2\lambda_{\text{cpa}}(\bar{q}-q^*)\left(\frac{G_{\text{ref}}}{G}\right)^{\theta_{\text{cpa}}} - \lambda_G = 0 \quad \ldots\ldots(162)$$

This is quadratic in q^* when $c_2 = 0$, cubic when $c_2 > 0$. The unconstrained solution (formal closed form when the polynomial degree allows) is written as

$$q^*_{\text{unc}}(t) = \frac{-\tilde{c}_1 + \sqrt{\tilde{c}_1^2 - 4\tilde{c}_2(\tilde{c}_0 - \lambda_G)}}{2\tilde{c}_2},\ldots\ldots(163)$$

where

$$\tilde{c}_0 = c_0 + w - 2\lambda_{\text{cpa}}\bar{q}\left(\frac{G}{G_{\text{ref}}}\right)^{\theta_{\text{cpa}}}, \tilde{c}_1 = c_1(1+G/G_{\text{ref}})^{\eta_q}, \tilde{c}_2 = c_2\left(\frac{G}{G_{\text{ref}}}\right)^{\eta_q'} \quad \ldots\ldots(164)$$

The constrained optimal control is the projection

$$q^*(t) = \text{proj}_{[0,q_{\max}]}\big(q^*_{\text{unc}}(t)\big)\ldots\ldots(165)$$

Optimal fertilizer F: set $\partial H/\partial F = 0$:

$$p_Y\frac{\partial Y_c}{\partial F} - c_{F,1} - c_{F,2}F^* - c_{F,\text{env}}(1+\epsilon_F)F^{*\epsilon_F} + \lambda_S(1+\psi_F F^*)^{2\phi_F} = 0 \quad \ldots\ldots(166)$$

Under standard shape conditions (concavity of Y_c in F and convexity of c_F), the implicit function theorem guarantees a unique $F^* \in [0,F_{\max}]$ if the second-derivative condition is

negative:

$$\frac{\partial^2 H}{\partial F^2} = p_Y \frac{\partial^2 Y_c}{\partial F^2} - c_{F,2} - c_{F,\text{env}}\epsilon_F(1+\epsilon_F)F^{\epsilon_F - 1} - 2\lambda_S \psi_F (1+\psi_F F)^{3\phi_F} < 0 \quad \ldots\ldots (167)$$

The controlled adoption rate and singular arcs

For the continuous adoption-rate control $\dot{A}_{\text{ctrl}} \in [-r_A, r_A]$, the Hamiltonian in this coordinate is quadratic:

$$H_A(\dot{A}_{\text{ctrl}}) = -c_A \dot{A}_{\text{ctrl}}^2 - c_{\text{learn}}(1-A)\dot{A}_{\text{ctrl}} + \lambda_A \dot{A}_{\text{ctrl}} \ldots\ldots (168)$$

The unconstrained maximizer is

$$\dot{A}_{\text{ctrl,unc}} = \frac{\lambda_A - c_{\text{learn}}(1-A)}{2c_A} \ldots\ldots (169)$$

Applying box constraints gives

$$\dot{A}^*_{\text{ctrl}} = \begin{cases} r_A, & \dot{A}_{\text{ctrl,unc}} \geq r_A, \\ \dot{A}_{\text{ctrl,unc}}, & |\dot{A}_{\text{ctrl,unc}}| < r_A \\ -r_A, & \dot{A}_{\text{ctrl,unc}} \leq -r_A. \end{cases} \ldots\ldots (170)$$

Two-point boundary value problem and shooting

The full optimality system is the two-point boundary value problem (TPBVP) (Zhao & Yan, 2025):

$$\begin{cases} \dot{x}^* = \nabla_\lambda H(x^*, u^*(x^*,\lambda), \lambda), \\ \dot{\lambda} = r\lambda - \nabla_x H(x^*, u^*(x^*,\lambda),\lambda), \\ x^*(0) = x_0, \lambda(T) = \nabla_x \Phi(x^*(T)). \end{cases} \ldots\ldots (171)$$

Solve via single shooting. Define the shooting function $S: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$S(\lambda_0) = \lambda(T; x_0, \lambda_0) - \nabla_x \Phi\big(x(T; x_0, \lambda_0)\big). \ldots\ldots (172)$$

Its Jacobian is

$$DS(\lambda_0) = \frac{\partial \lambda(T)}{\partial \lambda_0} - \nabla^2 \Phi\big(x(T)\big)\frac{\partial x(T)}{\partial \lambda_0} \ldots\ldots (173)$$

Connection to the MFG: decentralized optimality

The single-agent optimal control must be consistent with the mean-field equilibrium. Define the consistency (best-response) map $T: \mathcal{U}_{\text{ad}} \to \mathcal{U}_{\text{ad}}$ by

$$(T\bar{u})(t) = u^*(t; \bar{u}), \ldots\ldots \#(174)$$

where $u^*(\cdot; \bar{u})$ is the optimal control for the representative farmer when the aggregate control trajectory is $\bar{u}$. An MFG equilibrium is a fixed point $u^* = T(u^*)$.

Theorem 5.2. (contraction and uniqueness):

If the Lipschitz constant L_T of T satisfies $L_T < 1$ in the norm $\|u\|_{\mathcal{U}} = \text{ess sup}_{t\in[0,T]}\|u(t)\|_\infty$, then the MFG admits a unique Nash equilibrium and the iterated best-response sequence $u^{(n+1)} = T(u^{(n)})$ converges geometrically to u^*.

Spatial diffusion of innovation and social learning

The generalized cross-diffusion reaction-advection system

Let the technology adoption rate (Figure 4) $A(x,t) \in [0,1]$ evolve on the spatial domain $\Omega = [0,10] \times [0,10] \subset \mathbb{R}^2$. The dynamics follow a generalized Fisher-KPP equation augmented by cross-diffusion (Kogan, 2024), nonlocal social learning, and multiplicative stochastic forcing:

$$\partial_t A = \nabla \cdot \big(D_A(A,G)\nabla A + J_{\text{cross}}(A,G,S)\big) + F_A(A,G,S) + N(A;\kappa) + \sigma_A(A)\dot{W}_t^A, \ldots\ldots \#(175)$$

where $D_A \in C^1([0,1] \times \mathbb{R}_+; \mathbb{R}^{2\times 2})$ is a state-dependent diffusion tensor, J_{cross} is the cross-diffusion flux, F_A is the local reaction term, N is the nonlocal social learning operator, and $\dot{W}_t^A$ denotes spatiotemporal white noise.

The state-dependent diffusion tensor

The diffusion tensor incorporates density dependence, groundwater modulation, gradient-enhanced anisotropy, and wind alignment:

$$D_A(A,G) = D_0\left[I + \beta_1 A I + \beta_2\left(\frac{G_{\text{ref}}}{G}\right) I + \beta_3(\nabla A \otimes \nabla A) + \beta_4\left(\frac{G_{\text{crit}}}{G_{\text{crit}} - G}\right) + M(\theta_w)\right] \ldots\ldots (176)$$

with baseline diffusivity $D_0 = 0.15$, parameters $\beta_1 = 0.25$, $\beta_2 = 0.1$, $\beta_3 = 0.05$, $\beta_4 = 0.2$, and anisotropy matrix aligned with prevailing wind direction θ_w,

$$M(\theta_w) = \begin{pmatrix} \cos^2\theta_w & \sin\theta_w\cos\theta_w \\ \sin\theta_w\cos\theta_w & \sin^2\theta_w \end{pmatrix} \ldots\ldots (177)$$

The eigenvalues of D_A are

$$\lambda_\pm(D_A) = 2D_0[(1+\beta_1 A + \beta_2 G/G_{\text{ref}})(2+\beta_3|\nabla A|^2) + \beta_4\left(\frac{G_{\text{crit}}}{G_{\text{crit}} - G}\right) \pm \sqrt{\frac{\beta_4^2\left(\frac{G_{\text{crit}}}{G_{\text{crit}} - G}\right)^2 + 4\beta_3^2|\nabla A|^4}{4}}] \quad \ldots\ldots (178)$$

Assumption 6.1. (Uniform ellipticity).

There exists $\lambda_0 > 0$ such that for all $\xi \in \mathbb{R}^2$ and $(A,G) \in [0,1] \times \mathbb{R}_+$,

$$\xi^\top D_A(A,G)\,\xi \geq \lambda_0|\xi|^2 \ldots\ldots (179)$$

This holds provided $D_0 > 0$, $\beta_1, \beta_2, \beta_4 \geq 0$, and $\beta_3|\nabla A|^2 > -1$.

Cross-diffusion flux and Keller–Segel structure

The cross-diffusion flux couples adoption to groundwater and soil-carbon gradients:

$$J_{\text{cross}}(A,G,S) = \chi_1 A\nabla G + \chi_2 A(1-A)\nabla S + \chi_3 A(1-A)\nabla A \ldots\ldots (180)$$

with sensitivities $\chi_1 = 0.08$, $\chi_2 = 0.05$, and self-aggregation $\chi_3 = 0.02$.

This flux can be expressed in divergence form:

$$J_{\text{cross}} = \nabla \cdot (A\Gamma(G,S)) - \Gamma(G,S)\nabla A, \Gamma(G,S) = (\chi_1 G, \chi_2 S, \chi_3 A)^{\top} \quad \ldots\ldots (181)$$

Consequently, the equation admits a Keller-Segel-type form (Yasmin & Iqbal, 2022)

$$\partial_t A = \nabla \cdot \left(D_A^{\text{eff}}(A,G)\nabla A\right) - \nabla \cdot \left(A\nabla\varphi(G,S)\right) + F_A + N, \ldots\ldots (182)$$

where $\varphi(G,S) = \chi_1 G + \chi_2 S$ and the effective diffusion tensor is

$$D_A^{\text{eff}} = D_A + \chi_3 A(1-A)I \ldots\ldots (183)$$

The local reaction term

The local reaction generalizes Fisher–KPP logistic growth with groundwater-dependent carrying capacity and soil-carbon enhancement:

$$F_A(A,G,S) = r_A A(1-K_A A)\left(\frac{G_{\text{crit}}}{G_{\text{crit}} - G}\right) + \nu_A(1+\eta_A \frac{S_{\text{ref}}}{S})\,\varphi_S \exp(-2G_{\text{ref}}^2 \lambda_A (G - G_{\text{opt}})^2) \quad \ldots\ldots (184)$$

with $r_A = 0.08$, $K_A = 1$, $\nu_A = 1.5$, $\eta_A = 0.25$, $\varphi_S = 0.8$, $G_{\text{opt}} = 3$, and $\lambda_A = 0.5$.

An Allee-effect modification (Singh *et al*., 2025) is given by

$$F_A^{\text{Allee}}(A,G,S) = r_A A(K_A A - A_{\text{crit}})(1 - K_A A)\left(\frac{G_{\text{crit}}}{G_{\text{crit}} - G}\right) + \nu_A S(S) \ldots\ldots (185)$$

with critical threshold $A_{\text{crit}} = 0.05$.

Nonlocal social-learning kernel

The nonlocal operator $N: L^2(\Omega) \to L^2(\Omega)$ models social learning:

$$N(A)(x,t) = \int_\Omega \kappa\big(x,y;A(y,t)\big)[A(y,t) - A(x,t)]dy \quad \ldots\ldots (186)$$

where the kernel $\kappa: \Omega \times \Omega \times [0,1] \to \mathbb{R}_+$ is

$$\kappa(x,y;A) = \kappa_0 \exp(-\frac{2}{\sigma_{\text{soc}}^2}\|x-y\|^2)\Big[1 + \lambda_{\text{soc}}\,\mathbf{1}_{\{\|x-y\| \le d_{\text{village}}\}}\Big][1 + \mu_{\text{soc}} A(y,t)] \quad \ldots\ldots (187)$$

with $\kappa_0 = 0.25$, $\sigma_{\text{soc}} = 1.5$ (km), $\lambda_{\text{soc}} = 0.4$, $d_{\text{village}} = 2$ (km), and $\mu_{\text{soc}} = 0.3$. The kernel is symmetric: $\kappa(x,y;A) = \kappa(y,x;A)$.

Decomposing N into convolutional and local parts gives

$$N(A) = \frac{\kappa_0}{2\pi\sigma_{\text{soc}}^2}\big(G_{\sigma_{\text{soc}}} * A\big)(x) - \frac{\kappa_0}{2\pi\sigma_{\text{soc}}^2}A(x) + N_{\text{village}}(A) + N_{\text{peer}}(A), \ldots\ldots (188)$$

where G_σ is the Gaussian kernel and

$$N_{\text{village}}(A) = \lambda_{\text{soc}} \int_{B(x,d_{\text{village}})} \exp(-\frac{2}{\sigma_{\text{soc}}^2}\|x-y\|^2)[A(y) - A(x)]dy \ldots\ldots (189)$$

$$N_{\text{peer}}(A) = \mu_{\text{soc}} \int_\Omega \kappa_0 \exp(-\frac{2}{\sigma_{\text{soc}}^2}\|x-y\|^2)A(y)[A(y) - A(x)]dy \ldots\ldots (190)$$

Functional-analytic framework and well-posedness

Define the Gelfand triple for the adoption equation:

$$V_A \hookrightarrow H_A \hookrightarrow V_A',$$

with $H_A = L^2(\Omega)$ and $V_A = H^1(\Omega) \cap \{A: 0 \le A \le 1$. The bilinear form $a_A: V_A \times V_A \to \mathbb{R}$ is

$$a_A(A,\varphi) = \int_\Omega \big(D_A(A,G)\nabla A + J_{\text{cross}}(A,G,S)\big) \cdot \nabla\varphi\, dx - \int_\Omega F_A(A,G,S)\varphi\, dx - \int_\Omega N(A)\varphi\, dx \quad \ldots\ldots (191)$$

Weak formulation. Seek $A \in L^2(0,T;V_A) \cap L^\infty(0,T;H_A)$ such that for all $\varphi \in V_A$,

$$\langle \partial_t A, \varphi\rangle_{V_A' \times V_A} + a_A(A,\varphi) = 0 \quad \ldots\ldots (192)$$

Theorem 6.1 (Existence and uniqueness). Under Assumption 6.1, with initial data $A_0 \in H_A \cap [0,1]$, $G \in L^\infty(0,T;H^1(\Omega))$, and $S \in L^\infty(0,T;L^2(\Omega))$, there exists a unique weak solution $A \in C([0,T];H_A) \cap L^2(0,T;V_A)$, satisfying $0 \le A(x,t) \le 1$ for almost every $(x,t) \in \Omega \times [0,T]$.

Proof sketch. Use Galerkin approximations $A^{(n)}(t) = \sum_{j=1}^n \alpha_j^{(n)}(t)e_j$ with $\{e_j\}$ eigenfunctions of $-\Delta$ under Neumann boundary conditions. The coefficient ODEs are

$$\frac{d}{dt}\alpha_j^{(n)} = -\sum_{k=1}^n \alpha_k^{(n)} \int_\Omega D_A\big(A^{(n)},G\big)\nabla e_k \cdot \nabla e_j\, dx - \int_\Omega J_{\text{cross}} \cdot \nabla e_j\, dx + \int_\Omega F_A\big(A^{(n)},G,S\big)e_j\, dx + \int_\Omega N\big(A^{(n)}\big)e_j\, dx \ldots\ldots (193)$$

Local existence follows from Carathéodory's theorem (Fabila-Monroy & Huemer, 2017). The invariant set $\{0 \le A \le 1\}$ is preserved because $F_A(0,G,S) = F_A(1,G,S) = 0$ and $N(A)$ is a mass-conserving convolution that preserves bounds. Uniform a priori estimates give global existence; Lipschitz continuity of F_A and N on $[0,1]$ yields uniqueness.

Traveling wave analysis and front propagation

Reduce to one spatial dimension on $\mathbb{R}$ with constant $G = G_0 < G_{\text{crit}}$ and $S = S_0$. Seek a traveling wave $A(\xi)$

with $\xi = x - ct$ and speed $c > 0$. The traveling-wave equation is

$$-cA' = D_{\text{eff}}A'' + \chi A' + r_A A(1 - K_A A)G(G_0) + N_{1D}(A) \ldots\ldots (194)$$

where $D_{\text{eff}} = D_0(1 + \beta_1 A + \beta_2 G_0/G_{\text{ref}})$ and $\chi = \chi_1 G_0'$. The 1D nonlocal term is

$$N_{1D}(A)(\xi) = \int_{-\infty}^{\infty} \kappa_{1D}(\xi - \eta)[A(\eta) - A(\xi)]d\eta \ldots\ldots (195)$$

Linearizing about $A = 0$ ahead of the front gives

$$-cA' \approx D_0 A'' + r_A G(G_0)A + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}}\left(G_{\sigma_{\text{soc}}} * A - A\right) \ldots\ldots (196)$$

Seek exponential solutions $A(\xi) = e^{-\lambda\xi}$, $\lambda > 0$. This yields the dispersion relation

$$c\lambda = D_0\lambda^2 + r_A G(G_0) + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}}\left(e^{\sigma_{\text{soc}}^2\lambda^2/2} - 1\right) \ldots\ldots (197)$$

The minimal wave speed c^* is found by minimizing over $\lambda > 0$:

$$c^* = \min_{\lambda>0} \frac{D_0\lambda^2 + r_A G(G_0) + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}}(e^{\sigma_{\text{soc}}^2\lambda^2/2} - 1)}{\lambda} \ldots\ldots (198)$$

The tangency condition (derivative zero) gives

$$2D_0\lambda^* + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}}\sigma_{\text{soc}}^2\lambda^* e^{\sigma_{\text{soc}}^2\frac{(\lambda^*)^2}{2}} = \frac{D_0(\lambda^*)^2 + r_A G(G_0) + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}}(e^{\sigma_{\text{soc}}^2(\lambda^*)^2/2} - 1)}{\lambda^*} \ldots\ldots (199)$$

For small σ_{soc}, expanding to $O(\sigma_{\text{soc}}^2)$ yields the approximation

$$c^* \approx 2\sqrt{D_0 r_A G(G_0)}\,(1 + \frac{2D_0\kappa_0}{2\pi}\,\sigma_{\text{soc}}^3)^{\frac{1}{2}} \ldots\ldots (200)$$

Spectral stability of homogeneous steady states

Homogeneous steady states include $A^* = 0$, $A^* = K_A$, and possibly $A^* = A_{\text{crit}}$. Linearizing about $A^* = K_A$ with perturbations $A = K_A + \varepsilon e^{\sigma t + ik\cdot x}$ gives growth rate

$$\sigma(k) = -D_0|k|^2(1 + \beta_1 K_A + \beta_2 G_{\text{ref}}/G) - r_A K_A\, G(G)\, S(S) + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}^2}\left(e^{-|k|^2\sigma_{\text{soc}}^2/2} - 1\right) \ldots\ldots (201)$$

Stability (Reσ<0 for all $k \neq 0$) requires

$$r_A K_A\, G(G)\, S(S) < D_0|k|^2(1 + \beta_1 K_A + \beta_2 G_{\text{ref}}/G) + \frac{\kappa_0}{2\pi\sigma_{\text{soc}}^2}\left(1 - e^{-|k|^2\sigma_{\text{soc}}^2/2}\right) \ldots\ldots (202)$$

The longest wavelength (most dangerous) mode is $|k| \to 0$; since the right-hand side tends to zero as $|k| \to 0$, the fully saturated state $A = K_A$ is spectrally stable for all k when $G(G) > 0$.

Coupling to the mean-field game: the social-learning value

Adoption dynamics enter the mean-field game through the value function v and the population distribution m. The social learning term modifies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\partial_t v + H(x, \nabla v, m) + \int_\Omega \frac{\delta A}{\delta v}(x, y)N\big(A(y)\big)dy + \frac{1}{2}\text{tr}(a\nabla^2 v) = 0 \ldots\ldots (203)$$

Mean-field consistency requires the averages

$$\bar{A}(t) = \int_\Omega A(x,t)m(t,x)dx, \bar{N}(t) = \int_\Omega N(A)(x,t)m(t,x)dx \ldots\ldots (204)$$

The social-learning kernel creates a nonlocal game-theoretic coupling: decisions at x depend on the weighted neighborhood average of adoption, producing a spatial public-goods problem.

Discrete numerical realization

On the uniform grid $\{(x_i, y_j)\}_{i,j=0}^{49}$ with $\Delta x = \Delta y = 10/49$, an explicit time-stepping scheme is

$$A_{i,j}^{n+1} = A_{i,j}^n + \Delta t[\, D_A(\nabla_\Delta^2 A^n)_{i,j} + F_A(A_{i,j}^n, G_{i,j}^n) + N_\Delta\big(A^n\big)_{i,j}] \ldots\ldots (205)$$

where the five-point discrete Laplacian is

$$(\nabla_\Delta^2 A)_{i,j} = \frac{A_{i+1,j} + A_{i-1,j} + A_{i,j+1} + A_{i,j-1} - 4A_{i,j}}{\Delta x^2} \ldots\ldots (206)$$

and the discrete nonlocal operator is

$$N_\Delta(A)_{i,j} = \sum_{k,\ell} \kappa_{i,j,k,\ell}\,\big(A_{k,\ell} - A_{i,j}\big)\Delta x^2 \ldots\ldots (207)$$

with discrete kernel

$$\kappa_{i,j,k,\ell} = \kappa_0 \exp\left(-\frac{2}{\sigma_{\text{soc}}^2}\left(\big(x_i - x_k\big)^2 + \big(y_j - y_\ell\big)^2\right)\right)\Big[1 + \lambda_{\text{soc}}\mathbf{1}_{\{(x_i - x_k)^2 + (y_j - y_\ell)^2 \le d_{\text{village}}^2\}}\Big] \ldots\ldots (208)$$

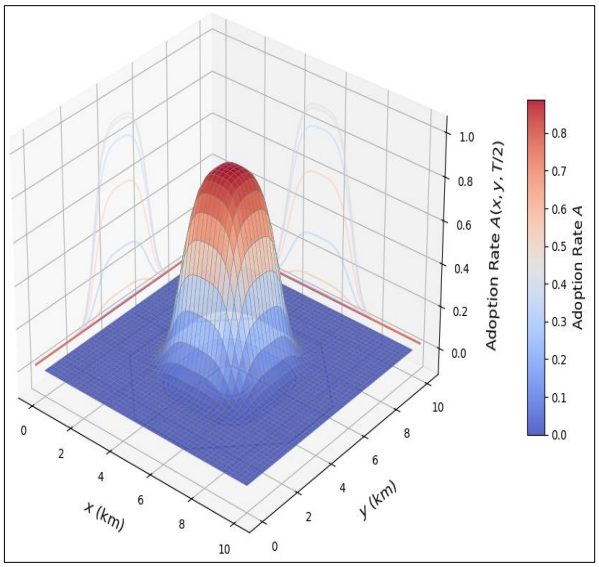


Figure 4: Technology adoption diffusion front: spatial mesh at t=T/2 with social learning topography

Coupled system dynamics and regime shift analysis

The deterministic skeleton and coupled ODE system

We consider the deterministic skeleton of the full coupled stochastic system by setting all noise terms to zero and restricting to spatially homogeneous states. The reduced state vector $z \in \mathbb{R}^6$ is

$$z(t) = \begin{pmatrix} G(t) \\ A(t) \\ S(t) \\ L(t) \\ \bar{q}(t) \\ \bar{A}(t) \end{pmatrix} \in Z_{\text{red}} = \mathbb{R}_+ \times\times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times\ldots\ldots(209)$$

The coupled deterministic dynamics are governed by

$$\frac{d}{dt}z = F(z;\mu) = \begin{pmatrix} -\kappa\bar{q} + R(G) - EE^{\mathrm{T}}(G) r_A\, A(1-\frac{A}{K_A})\, G(G)\, S(S) + \bar{N}(A) + \dot{A}^*_{\text{ctrl}} \\ DS_{\text{eff}}(S_{\text{in}} - S) - k_{\text{dec}} S e^{-E_a/(R_g T_{\text{soil}})} \\ r_L\, L(1-\frac{L}{K_L(G,S)}) - \Psi_q(G, \partial_G v^*; w)\, \bar{q} - \Psi_A(A, \partial_A v^*; c_{\text{fix}})\, \bar{A} \\ (\textit{remaining components arranged consistently as above}) \end{pmatrix} \ldots\ldots(210)$$

where Ψ_q and Ψ_A are the mean-field consistency maps for extraction and adoption, respectively, and $\mu = (w, c_{\text{fix}}, r, \lambda_\tau, \kappa, r_A, r_L, k_{\text{dec}}, \ldots)^{\mathrm{T}} \in \mathbb{R}^p$ is the bifurcation-parameter vector.

Equilibrium manifold and fixed-point analysis

The equilibrium manifold $M_{\text{eq}} \subset Z_{\text{red}} \times \mathbb{R}^p$ is

$$M_{\text{eq}} = \{(z^*, \mu) \in Z_{\text{red}} \times \mathbb{R}^p : F(z^*;\mu) = 0\}\ldots\ldots\ldots\ldots(211)$$

The fixed-point equations decompose component-wise. Writing equilibrium values with a star superscript:

Groundwater equilibrium:

$$0 = -\kappa\bar{q}^* + n_e R_{\text{eff}}(1-\frac{G_{\max}}{G^*}) + \gamma_R - E_0(1+\kappa_T(T_{\text{air}} - T_{\text{ref}}))(\frac{G_{\text{crit}}}{G_{\text{crit}}} - \frac{G^*}{}) + \delta_E\ldots\ldots(212)$$

Adoption equilibrium:

$$0 = r_A A^*(1-\frac{A^*}{K_A})(\frac{G_{\text{crit}}}{G_{\text{crit}}} - \frac{G^*}{}) + \nu_A(1+\eta_A\frac{S^*}{S_{\text{ref}}}) + \bar{N}(A^*) + \dot{A}^*_{\text{ctrl}}(A^*, \lambda^*_A)\ldots\ldots(213)$$

Soil-carbon equilibrium:

$$0 = DS_{\text{eff}}(S_{\text{in}} - S^*) - k_{\text{dec}} S^* e^{-E_a/(R_g T_{\text{soil}})} + 1 + \psi_F F^* \phi_F(F^*) + \phi_A I^*($$

Livestock equilibrium:

$$0 = r_L L^*(1-\frac{L^*}{K_L(G^*, S^*)})\ldots\ldots(215)$$

Mean-field consistency:

$$\bar{q}^* = q^*(G^*, \lambda^*_G; w), \bar{A}^* = A^*\ldots\ldots(216)$$

Linearization and spectral stability

The Jacobian matrix $J(z;\mu) = DF|_z \in \mathbb{R}^{6\times 6}$ evaluated at an equilibrium z^* is

$$J(z^*;\mu) = \begin{pmatrix} \partial_G\dot{G} & \partial_A\dot{G} & \partial_S\dot{G} & \partial_L\dot{G} & \partial_{\bar{q}}\dot{G} & \partial_{\bar{A}}\dot{G} \\ \partial_G\dot{A} & \partial_A\dot{A} & \partial_S\dot{A} & \partial_L\dot{A} & \partial_{\bar{q}}\dot{A} & \partial_{\bar{A}}\dot{A} \\ \partial_G\dot{S} & \partial_A\dot{S} & \partial_S\dot{S} & \partial_L\dot{S} & \partial_{\bar{q}}\dot{S} & \partial_{\bar{A}}\dot{S} \\ \partial_G\dot{L} & \partial_A\dot{L} & \partial_S\dot{L} & \partial_L\dot{L} & \partial_{\bar{q}}\dot{L} & \partial_{\bar{A}}\dot{L} \\ \partial_G\dot{\bar{q}} & \partial_A\dot{\bar{q}} & \partial_S\dot{\bar{q}} & \partial_L\dot{\bar{q}} & \partial_{\bar{q}}\dot{\bar{q}} & \partial_{\bar{A}}\dot{\bar{q}} \\ \partial_G\dot{\bar{A}} & \partial_A\dot{\bar{A}} & \partial_S\dot{\bar{A}} & \partial_L\dot{\bar{A}} & \partial_{\bar{q}}\dot{\bar{A}} & \partial_{\bar{A}}\dot{\bar{A}} \end{pmatrix}_{z=z^*} \ldots\ldots(217)$$

The characteristic polynomial is

$$P(\lambda;\mu) = \det(J - \lambda I_6) = \lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0\ldots\ldots(218)$$

The coefficients a_k for $k = 1, \ldots, 6$ may be obtained via the Faddeev–LeVerrier algorithm (Baer, 2021):

$$a_1 = -\mathrm{tr}(J)\ldots\ldots(219)$$

$$a_2 = -\frac{1}{2}(a_1\mathrm{tr}(J) + \mathrm{tr}(J^2))\ldots\ldots(220)$$

$$a_3 = -\frac{1}{3}(a_2\mathrm{tr}(J) + a_1\mathrm{tr}(J^2) + \mathrm{tr}(J^3))\ldots\ldots(221)$$

and for $k = 4,5,6$,

$$a_k = -\frac{1}{k}\sum_{j=1}^{k} a_{k-j}\mathrm{tr}(J^j), a_0 = 1\ldots\ldots(222)$$

The Routh–Hurwitz stability criterion (Yang *et al*., 2022) gives necessary and sufficient conditions for local asymptotic stability of the equilibrium z^*. Define the

leading principal minors $\Delta_1, \ldots, \Delta_6$ of the Hurwitz matrix constructed from $\{a_k\}$. The required positivity conditions are

$$\Delta_1 = a_1 > 0, \ldots\ldots (223)$$

$$\Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0, \ldots\ldots (224)$$

$$\Delta_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \ldots\ldots (225)$$

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & a_6 & a_5 & a_4 \end{vmatrix} > 0, \ldots\ldots (226)$$

$$\Delta_5 = \begin{vmatrix} a_1 & 1 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & a_6 & a_5 \end{vmatrix} > 0, \ldots\ldots (227)$$

$$\Delta_6 = a_6 \Delta_5 > 0 \ldots\ldots (228)$$

Hopf Bifurcation and Limit Cycle Emergence

A Hopf bifurcation (Delmeire *et al*., 2025) occurs when a complex-conjugate pair of eigenvalues crosses the imaginary axis. The basic conditions are

$$\mathrm{tr}(J) = 0, \det(J) > 0, \frac{d}{d\mu} \mathrm{Re}(\lambda_{1,2}(\mu))\Big|_{\mu=\mu_c} \neq 0 \ldots\ldots (229)$$

Specializing to the parameter $\mu = w$ (groundwater price), the Hopf conditions are:

Pure imaginary eigenvalues:

$$\lambda_{1,2}(\mu_c) = \pm i\omega_0, \omega_0 > 0 \ldots\ldots (230)$$

Transversality:

$$\frac{d}{d\mu} \mathrm{Re}(\lambda_{1,2}(\mu))\Big|_{\mu=\mu_c} \neq 0 \ldots\ldots (231)$$

Nondegeneracy: the first Lyapunov coefficient (Badiale & Cravero, 2026) $l_1(\mu_c) \neq 0$.

The eigenvalue perturbation formula used to verify transversality is

$$\frac{d\mu}{d\lambda} = \frac{w^{\mathrm{T}} v}{w^{\mathrm{T}} \partial_\mu J \, v}, \ldots\ldots (232)$$

where v and w are the right and left eigenvectors corresponding to $\lambda = i\omega_0$.

The Hopf bifurcation theorem (for $F \in C^3$) states that under these conditions there exists a one-parameter family of periodic solutions $z(t;\varepsilon)$ with period

$$T(\varepsilon) = \frac{2\pi}{\omega_0} + O(\varepsilon^2), \ldots\ldots (233)$$

and amplitude

$$\|z(t;\varepsilon) - z^*\| = \varepsilon + O(\varepsilon^2),$$

bifurcating from z^* for μ near μ_c. The bifurcation is supercritical if $l_1(\mu_c) < 0$ and subcritical if $l_1(\mu_c) > 0$.

Center manifold reduction

Near the Hopf point μ_c the dynamics reduce to a two-dimensional center manifold $W_c \subset Z_{\mathrm{red}}$. Let v_1, v_2 be the real and imaginary parts of the critical eigenvector; the center subspace is $E_c = \mathrm{span}\{v_1, v_2\}$.

Parameterize the center manifold by $u \in \mathbb{C}$ with $u = \langle w, z - z^* \rangle$, where w is the adjoint eigenvector. The reduced dynamics are

$$\dot{u} = \lambda(\mu)u + g(u, \bar{u}; \mu), \ldots\ldots (234)$$

where g contains nonlinear terms. Expanded to cubic order,

$$\dot{u} = i\omega_0 u + c_1 u^2 \bar{u} + O(|u|^4) \ldots\ldots (235)$$

The first Lyapunov coefficient at μ_c can be written

$$l_1(\mu_c) = \frac{1}{2\omega_0^2} \mathrm{Re}(i\omega_0 c_1 + \overline{c_1} c_2), \ldots\ldots (236)$$

where c_1, c_2 are computed from the second and third derivatives of F projected onto E_c.

Saddle-node and trans-critical bifurcations

Saddle-node bifurcations occur when two equilibria collide and annihilate. The condition is

$$\det\big(J(z^*;\mu)\big) = 0, \partial_\mu \det\big(J(z^*;\mu)\big) \neq 0 \ldots\ldots (237)$$

Near a saddle-node the one-dimensional normal form on the center manifold is

$$\dot{u} = \mu - u^2 + O(u^3) \ldots\ldots \#(238)$$

For the groundwater–adoption system (Figure 5), a trans-critical bifurcation may occur when the adoption-free equilibrium $A^* = 0$ exchanges stability with the adoption-saturated equilibrium $A^* = K_A$. A representative condition at $A = 0$ is

$$\frac{\partial \dot{A}}{\partial A}\Big|_{A=0} = 0, \ldots\ldots (239)$$

which, expanded using model terms, becomes

$$0 = r_A G(G^*) S(S^*) + \frac{\partial \bar{N}}{\partial A}\Big|_{A=0} + \frac{\partial \dot{A}^*_{\mathrm{ctrl}}}{\partial A}\Big|_{A=0} \ldots\ldots (240)$$

Cusp bifurcation and regime-shift hysteresis

A cusp bifurcation arises when two saddle-node curves meet tangentially, producing a region of bi-stability. The normal form is

$$\dot{u} = \alpha + \beta u - u^3 \ldots\ldots (241)$$

At the cusp point (α_c, β_c) the conditions are

$$\alpha_c = 0, \beta_c = 0, \frac{\partial}{\partial \beta} \frac{\partial^2 \dot{u}}{\partial u^2} \neq 0, \frac{\partial^3 \dot{u}}{\partial u^3} \neq 0 \ldots\ldots (242)$$

In the (w, c_{fix}) parameter plane the cusp implies hysteresis: for intermediate w there may exist two stable equilibria (high-adoption/low-extraction and low-adoption/high-extraction) separated by an unstable saddle. The critical transition surface is

$$\Sigma_{\text{crit}} = \{(w, c_{\text{fix}}) \in \mathbb{R}^2_+ : \exists z_1^* \neq z_2^* \ldots\ldots (243)$$

Lyapunov exponents of the stochastic system

For the full stochastic system the top Lyapunov exponent characterizes regime stability. Consider the linearized stochastic flow $\Phi(t, \omega)$ satisfying the Stratonovich SDE

$$d\Phi = J(z(t;\omega))\,\Phi\,dt + \nabla\Sigma(z(t;\omega))\,\Phi \circ dW_t \ldots\ldots (244)$$

By the multiplicative ergodic theorem the Lyapunov exponents

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\Phi(t,\omega)v_i\|}{\|v_i\|}, i = 1, \ldots, 6, \ldots\ldots (245)$$

exist almost surely for appropriate initial vectors v_i. The top exponent λ_1 determines stability:

$$\lambda_1 < 0 \Rightarrow \text{stochastic stability}, \lambda_1 > 0 \Rightarrow \text{stochastic chaos} \ldots\ldots (246)$$

The Lyapunov (Kaplan–Yorke) dimension is

$$D_{\text{KY}} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \ldots\ldots (247)$$

where j is the largest integer such that $\sum_{i=1}^{j} \lambda_i \geq 0$.

Basin of attraction and critical slowing down

The basin of attraction $B(z^*)$ of a stable equilibrium is

$$B(z^*) = \{z_0 \in Z_{\text{red}} : \lim_{t\to\infty} \varphi(t; z_0) = z^*\}, \ldots\ldots (248)$$

where $\varphi(t; z_0)$ denotes the deterministic flow. The basin boundary ∂B is formed by the stable manifold of the relevant saddle equilibrium.

The return time to equilibrium diverges near a bifurcation, exhibiting critical slowing down:

$$\tau_{\text{return}} \sim |\mu - \mu_c|^{-\nu}, \ldots\ldots (249)$$

with critical exponent $\nu = \frac{1}{2}$ for a saddle-node and $\nu = 1$ for a Hopf bifurcation. Consequently, the variance of observables increases near the bifurcation. For example,

$$\text{Var}(G) \sim \frac{2\,\sigma_{\text{eff}}^2}{|\lambda_{\text{dom}}|} \sim |\mu - \mu_c|^{-1}, \ldots\ldots (250)$$

where λ_{dom} is the dominant eigenvalue approaching zero and σ_{eff} is an effective noise amplitude.

Spatiotemporal bifurcation in the full PDE system

For the spatially extended (PDE) model, linearize about a homogeneous equilibrium (G^*, A^*, S^*). The linear operator takes the block form

$$\mathcal{L} = \begin{pmatrix} D_G\nabla^2 + \partial_G\dot{G} & \partial_A\dot{G} & \partial_S\dot{G} \\ \partial_G\dot{A} & D_A\nabla^2 + \partial_A\dot{A} & \partial_S\dot{A} \\ \partial_G\dot{S} & \partial_A\dot{S} & D_S\nabla^2 + \partial_S\dot{S} \end{pmatrix}, \ldots\ldots (251)$$

where D_G, D_A, D_S are diffusion coefficients and partial derivatives are evaluated at the homogeneous equilibrium.

Seek normal modes of the form $z(x,t) = z^* + \hat{z}\, e^{\sigma t + ik\cdot x}$. This yields the dispersion relation

$$\det(\mathcal{L}(ik) - \sigma I) = 0, \ldots\ldots (252)$$

where $\mathcal{L}(ik)$ is the Fourier symbol obtained by replacing ∇^2 by $-|k|^2$.

A Turing bifurcation occurs when $\text{Re}(\sigma(k_c)) = 0$ for some critical wavenumber $k_c \neq 0$ while $\text{Re}(\sigma(0)) < 0$. Representative Turing-type conditions are

$$\text{tr}(\mathcal{L}(0)) < 0, \ldots\ldots (253)$$

$$\det(\mathcal{L}(0)) < 0, \ldots\ldots (254)$$

and the diffusion-driven instability inequality (schematic form)

$$D_A\,\partial_G\dot{G} + D_G\,\partial_A\dot{A} > 2D_G D_A \det(\mathcal{L}(0)) \ldots\ldots (255)$$

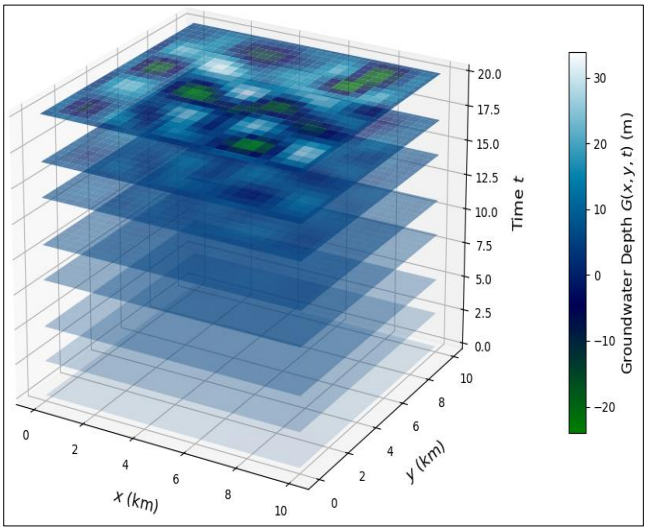


Figure 5: Spatiotemporal groundwater depletion volume: (x, y, t) ↦ G (x, y, t)

Bayesian learning and adaptive expectations under climate uncertainty

The climate state-space model and incomplete information

We consider a representative farmer who has incomplete information about the true climate state $\theta_t \in \Theta \subset \mathbb{R}^q$, which evolves as a hidden Markov process. The farmer observes a noisy signal $Y_t \in \mathbb{R}^m$ produced by the biophysical system. The complete state-space model is

$$d\theta_t = b_\theta(\theta_t)\,dt + \sigma_\theta(\theta_t)\,dW_t^\theta, \theta_0 \sim \pi_0 \ldots\ldots (256)$$

$$dY_t = h(\theta_t, x_t, u_t)\, dt + \sigma_Y(x_t, u_t)\, dV_t \ldots\ldots (257)$$

where $W_t^\theta \in \mathbb{R}^q$ and $V_t \in \mathbb{R}^m$ are independent standard Wiener processes on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with $W_t^\theta \perp V_t$. The drift $b_\theta: \Theta \to \mathbb{R}^q$ captures mean-reverting climate dynamics, $\sigma_\theta: \Theta \to \mathbb{R}^{q\times q}$ is the climate volatility matrix, $h: \Theta \times X \times U \to \mathbb{R}^m$ is the observation function, and $\sigma_Y: X \times U \to \mathbb{R}^{m\times m}$ is the observation noise matrix.

In the agricultural application the climate state decomposes as $\theta_t = (\theta_t^R, \theta_t^T, \theta_t^C)^\top$, representing mean rainfall, temperature trend, and a climate variability index. The observation vector is $Y_t = (Y_t^{\text{obs}}, ET_t^{\text{obs}}, G_t^{\text{obs}})^\top$, comprising observed crop yield, evapotranspiration, and groundwater depth.

The filtering problem and posterior measure

The farmer's information at time t is the observation filtration $\mathcal{Y}_t = \sigma(Y_s: 0 \leq s \leq t)$. The objective is to characterize the conditional distribution of the climate state given observations:

$$\pi_t(d\theta) = \mathbb{P}(\theta_t \in d\theta \mid \mathcal{Y}_t) \in \mathcal{P}(\Theta) \ldots\ldots (258)$$

The filter π_t is a measure-valued stochastic process $\pi_t: [0,T] \to \mathcal{P}(\Theta)$ satisfying a stochastic evolution equation on the space of probability measures. For any test function $\varphi \in C_b^2(\Theta)$ the filtered estimate is

$$\hat{\varphi}_t = \int_\Theta \varphi(\theta)\, \pi_t(d\theta) = \mathbb{E}[\varphi(\theta_t) \mid \mathcal{Y}_t] \ldots\ldots (259)$$

Define the conditional expectation of the observation function

$$\hat{h}_t = \int_\Theta h(\theta, x_t, u_t)\, \pi_t(d\theta) \ldots\ldots (260)$$

The innovation process, representing the surprise in new observations, is

$$\begin{aligned} dI_t = \sigma_Y^{-1}(x_t, u_t)[dY_t - \hat{h}_t\, dt] \\ = \sigma_Y^{-1}(x_t, u_t)[h(\theta_t, x_t, u_t) \\ - \hat{h}_t]\, dt + dV_t \ldots\ldots (261) \end{aligned}$$

By Lévy's characterization (Hu *et al*., 2019), I_t is a standard $\mathcal{Y}_t$-Wiener process.

The Kalman-Bucy filter: linear-Gaussian case

When the climate dynamics and observation function are linear and the initial prior is Gaussian, the posterior remains Gaussian for all time. Suppose

$$\begin{aligned} b_\theta(\theta) = -\Lambda(\theta - \theta^{\text{eq}}), \sigma_\theta(\theta) = \Sigma_\theta, h(\theta, x, u) \\ = H(x,u)\theta + h_0(x,u) \ldots\ldots (262) \end{aligned}$$

with $\Lambda \in \mathbb{R}^{q\times q}$ positive definite, $\Sigma_\theta \in \mathbb{R}^{q\times q}$, and $H(x,u) \in \mathbb{R}^{m\times q}$. Let the prior be $\pi_0 = \mathcal{N}(\hat{\theta}^0, P_0)$.

Theorem 8.1 (Kalman–Bucy). The posterior $\pi_t = \mathcal{N}(\hat{\theta}_t, P_t)$ evolves according to

$$\begin{aligned} d\hat{\theta}_t = -\Lambda(\hat{\theta}_t - \theta^{\text{eq}})\, dt + P_t H^\top(x_t, u_t) R_Y^{-1}(x_t, u_t)[dY_t - H(x_t, u_t)\hat{\theta}_t \, d \\ \frac{d}{dt} P_t \\ = -\Lambda P_t - P_t \Lambda^\top + Q_\theta \\ - P_t H^\top(x_t, u_t) R_Y^{-1}(x_t, u_t) H(x_t, u_t) P_t \ldots\ldots (264) \end{aligned}$$

where $Q_\theta = \Sigma_\theta \Sigma_\theta^\top$ is the climate process covariance and $R_Y = \sigma_Y \sigma_Y^\top$ is the observation noise covariance. The Riccati equation (Rivera-Oliva, 2025) for the posterior covariance admits a steady-state solution P^{ss} satisfying the algebraic Riccati equation

$$0 = -\Lambda P^{\text{ss}} - P^{\text{ss}} \Lambda^\top + Q_\theta - P^{\text{ss}} H^\top R_Y^{-1} H P^{\text{ss}} \ldots\ldots (265)$$

Proposition 8.1 (Convergence). If the pair (H, Λ) is observable and (Λ, Σ_θ) is controllable, then $P_t \to P^{\text{ss}}$ exponentially as $t \to \infty$, and the steady-state filter is asymptotically stable.

The steady-state Kalman gain is

$$K^{\text{ss}} = P^{\text{ss}} H^\top R_Y^{-1} \ldots\ldots (266)$$

and the steady-state innovation covariance is

$$S^{\text{ss}} = H P^{\text{ss}} H^\top + R_Y \ldots\ldots (267)$$

The Kushner-Stratonovich equation: nonlinear filtering

For the general nonlinear case, when the posterior has a density $\pi_t(\theta)$ with respect to Lebesgue measure, it evolves according to the Kushner-Stratonovich equation (Campillo & Corso, 2026):

$$\begin{aligned} d\pi_t(\theta) = \mathcal{L}_\theta^* \pi_t(\theta)\, dt + \pi_t(\theta)[h(\theta, x_t, u_t) \\ - \hat{h}_t]^\top R_Y^{-1}(x_t, u_t)[dY_t \\ - \hat{h}_t\, dt] \ldots\ldots (268) \end{aligned}$$

where $\mathcal{L}_\theta^*$ is the Fokker–Planck (adjoint) operator of the climate state:

$$\begin{aligned} \mathcal{L}_\theta^* \pi = -\nabla_\theta \cdot (\pi\, b_\theta) + \frac{1}{2} \nabla_\theta \cdot \nabla_\theta \cdot (\pi\, a_\theta), a_\theta \\ = \sigma_\theta \sigma_\theta^\top \ldots\ldots (269) \end{aligned}$$

In Stratonovich form the equation can be written as

$$\begin{aligned} \partial_t \pi_t = \mathcal{L}_\theta^* \pi_t + \pi_t [h - \hat{h}_t]^\top R_Y^{-1} \circ (\dot{Y}_t - \frac{1}{2}(h \\ + \hat{h}_t)) \ldots\ldots (270) \end{aligned}$$

Normalization of the posterior is preserved:

$$\int_\Theta \pi_t(\theta)\, d\theta = 1, \forall t \geq 0 \ldots\ldots (271)$$

The Zakai equation and unnormalized filtering

The Zakai equation gives an unnormalized linear equation for the posterior. Define the unnormalized density $\rho_t(\theta)$ through the Kallianpur-Striebel relation (Qiao, 2020)

$$\pi_t(\theta) = \frac{\rho_t(\theta)}{\int_\Theta \rho_t(\theta')\, d\theta'} \ldots\ldots (272)$$

The Zakai equation is

$$d\rho_t(\theta) = \mathcal{L}_\theta^* \rho_t(\theta)\, dt + \rho_t(\theta)\, h(\theta, x_t, u_t)^\top R_Y^{-1}(x_t, u_t)\, dY_t \ldots\ldots (273)$$

This is a linear stochastic partial differential equation on $L^2(\Theta)$. The mild solution can be written as

$$\rho_t = T_t \rho_0 + \int_0^t T_{t-s}(\rho_s\, h^\top R_Y^{-1})\, dY_s, \ldots\ldots (274)$$

where $\{T_t\}_{t\geq 0}$ is the semigroup generated by $\mathcal{L}_\theta^*$.

Adaptive expectations and the learning premium

Under incomplete information the farmer replaces the true climate state with its filtered estimate. The effective (adaptive) Hamiltonian becomes

$$H^{\text{adaptive}}(x, \nabla v, \pi_t) = \sup_{u\in U}\{e^{-rt}\int_\Theta \Pi(x,u,\theta)\, \pi_t(d\theta) + \nabla v \cdot \int_\Theta b(x,u,\theta)\, \pi_t(d\theta)\} \ldots\ldots \#(275)$$

Because the observation function h depends nonlinearly on θ and the observation noise may depend on control, the separation principle does not hold. The value function $v(t,x,\pi)$ therefore depends on the infinite-dimensional belief $\pi \in \mathcal{P}(\Theta)$.

Define the learning premium (expected value of perfect information, EVPI) by

$$P_{\text{learn}}(t,x) = v^{\text{full}}(t,x,\theta_t) - \int_\Theta v^{\text{partial}}(t,x,\theta)\, \pi_t(d\theta), \ldots\ldots (276)$$

where v^{full} is the value under full information and v^{partial} is the value given the current belief. By Jensen's inequality and concavity of the value function in θ,

$$P_{\text{learn}}(t,x) \geq \frac{1}{2}\text{tr}\,(P_t\, \nabla^2_{\theta\theta} v^{\text{partial}}(t,x,\hat{\theta}_t)) + O(\|P_t\|^2) \ldots\ldots (277)$$

Particle filtering and sequential Monte Carlo

For numerical implementation we may use a particle filter with N_p particles $\{\theta_t^{(i)}\}_{i=1}^{N_p}$ and weights $\{w_t^{(i)}\}_{i=1}^{N_p}$. The posterior is approximated as

$$\pi_t^N(d\theta) = \sum_{i=1}^{N_p} w_t^{(i)} \delta_{\theta_t^{(i)}}(d\theta), \sum_{i=1}^{N_p} w_t^{(i)} = 1 \ldots\ldots (278)$$

A Sequential Importance Resampling (SIR) implementation proceeds as follows.

Prediction:

$$\theta_{t+\Delta t}^{(i)} = \theta_t^{(i)} + b_\theta(\theta_t^{(i)})\Delta t + \sigma_\theta(\theta_t^{(i)})\sqrt{\Delta t}\, \xi^{(i)}, \ldots\ldots (279)$$

with $\xi^{(i)}$ standard normal draws.

Update (unnormalized weights):

$$\widetilde{w}_{t+\Delta t}^{(i)} = w_t^{(i)} \exp\left(-\frac{1}{2}\|Y_{t+\Delta t} - Y_t - h(\theta_t^{(i)}, x_t, u_t)\Delta t\|^2_{R_Y^{-1}}\right), \ldots\ldots (280)$$

where $\|z\|^2_{R_Y^{-1}} = z^\top R_Y^{-1} z$.

Normalization:

$$w_{t+\Delta t}^{(i)} = \frac{\widetilde{w}_{t+\Delta t}^{(i)}}{\sum_{j=1}^{N_p} \widetilde{w}_{t+\Delta t}^{(j)}} \ldots\ldots (281)$$

Resampling: Compute the effective sample size

$$N_{\text{eff}} = \frac{1}{\sum_{i=1}^{N_p} (w_{t+\Delta t}^{(i)})^2} \ldots\ldots (282)$$

If $N_{\text{eff}} < N_{\text{thresh}}$, perform systematic resampling.

The effective sample size monitors filter degeneracy.

Mean-field game with learning: the McKean–Vlasov filter

In a mean-field game as the number of farmers $N \to \infty$, each farmer i maintains a private filter π_t^i. The empirical measure of beliefs is

$$M_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\pi_t^i} \in \mathcal{P}(\mathcal{P}(\Theta)) \ldots\ldots (283)$$

The mean-field limit $M_t^N \rightharpoonup M_t \in \mathcal{P}(\mathcal{P}(\Theta))$ describes the distribution of beliefs across the population. The coupled system for a representative agent becomes

$$d\theta_t = b_\theta(\theta_t)\, dt + \sigma_\theta(\theta_t)\, dW_t^\theta, \ldots\ldots (284)$$

$$dY_t^i = h(\theta_t, x_t^i, u_t^i)\, dt + \sigma_Y(x_t^i, u_t^i)\, dV_t^i, \ldots\ldots (285)$$

$$d\pi_t^i = \mathcal{L}_\theta^* \pi_t^i\, dt + \pi_t^i[h(\cdot, x_t^i, u_t^i) - \hat{h}_t^i]^\top R_Y^{-1}[dY_t^i - \hat{h}_t^i\, dt], \ldots\ldots (286)$$

$$u_t^{i,*} = \arg\sup_{u\in U} H^{\text{adaptive}}(x_t^i, \nabla v(t, x_t^i, \pi_t^i), \pi_t^i) \ldots\ldots (287)$$

Mean-field consistency for beliefs requires

$$M_t = \text{Law}(\pi_t^i \mid M_t, \{u_j^*\}_{j\in J}) \ldots\ldots (288)$$

Convergence of beliefs and information cascades

Posterior contraction rates are governed by a Bernstein-von Mises type theorem (Borgert & Hannig, 2026). Under regularity conditions, as $t \to \infty$,

$$\sqrt{t}\,(\hat{\theta}_t - \theta^*) \xrightarrow{d} \mathcal{N}(0, I_{\text{Fisher}}^{-1}(\theta^*)), \ldots\ldots (289)$$

where the Fisher information matrix is

$$I_{\text{Fisher}}(\theta) = \mathbb{E}[\nabla_\theta h\, R_Y^{-1}\, \nabla_\theta h^\top] \ldots\ldots (290)$$

The asymptotic posterior covariance satisfies

$$\lim_{t\to\infty} t\, P_t = I_{\text{Fisher}}^{-1}(\theta^*) \ldots\ldots (291)$$

An information cascade can occur when social signals

dominate private signals. A representative threshold condition can be expressed as

$$\lambda_{\min}(R_Y)\,\kappa_0\,\sigma_{\text{soc}}^2 > 1, \ldots\ldots (292)$$

in which regime beliefs may converge to a consensus that is not aligned with the true climate state, producing herding and potential welfare losses.

The dual control problem: active learning

A farmer can choose controls that trade off immediate profit against information gain. The expected Fisher information accumulated up to time t under control path u is

$$I_t(u) = \mathbb{E}[\int_0^t \nabla_\theta h(\theta_s, x_s, u_s)^\top R_Y^{-1}(x_s, u_s) \nabla_\theta h(\theta_s, x_s, u_s)\, ds \mid \mathcal{Y}_t] \ldots\ldots (293)$$

The dual control objective augments expected profit with an information premium:

$$\max_u \mathbb{E}[\int_0^T e^{-rs}(\Pi(x_s, u_s, \theta_s) + \lambda_{\text{info}} \text{tr}(I_s(u)))\, ds + e^{-rT}\Phi(x_T)], \ldots\ldots (294)$$

where $\lambda_{\text{info}} > 0$ is the shadow price of information.

The resulting HJB-belief equation for the value $v(t, x, \pi)$ takes the form

$$\partial_t v + \mathcal{L}_\pi v + \sup_{u \in U}\{e^{-rt}\Pi(x, u, \pi) + \nabla_x v \cdot b(x, u, \pi) + \lambda_{\text{info}} \text{tr}(J_{\text{info}})\} = 0 \ldots\ldots (295)$$

where $\mathcal{L}_\pi$ is the generator of the belief process (the Kushner operator) and J_{info} denotes the Jacobian (or relevant information gain term) of the observation function with respect to θ.

Scenario ensemble analysis and sensitivity decomposition

The Monte Carlo ensemble framework

We construct a scenario ensemble of M independent realizations of the coupled stochastic dynamical system over the parameter space $\Theta \subset \mathbb{R}^{45}$. Each ensemble member $m \in \{1, \ldots, M\}$ corresponds to a sampled parameter vector $\theta^{(m)} \sim \pi_\Theta$ drawn from the prior distribution on parameters, and a realization of the driving noise processes $\{W_t^{(m)}\}_{t \in [0,T]}$. The ensemble state is

$$z^{(m)}(t; \theta^{(m)}, \omega^{(m)}) = (G^{(m)}(t), A^{(m)}(t), S^{(m)}(t), L^{(m)}(t), m^{(m)}(t), v^{(m)}(t)) \in \mathcal{Z}, m = 1, \ldots, M \ldots\ldots (296)$$

The quantity of interest (QoI) is a functional $J: \mathcal{Z} \times [0, T] \to \mathbb{R}$ representing aggregate system performance, for example cumulative discounted profit:

$$J^{(m)} = \int_0^T e^{-rt}\,\Pi(z^{(m)}(t), u^{*,(m)}(t))\, dt + e^{-rT}\,\Phi(z^{(m)}(T)) \ldots\ldots (297)$$

The Monte Carlo estimator of the expected QoI is

$$\widehat{\mathbb{E}}^M[J] = \frac{1}{M}\sum_{m=1}^M J^{(m)} \ldots\ldots (298)$$

with sample variance

$$\widehat{\text{Var}}^M(J) = \frac{1}{M-1}\sum_{m=1}^M (J^{(m)} - \widehat{\mathbb{E}}^M[J])^2 \ldots\ldots (299)$$

and standard error of the mean

$$\text{SEM} = \sqrt{\frac{\widehat{\text{Var}}^M(J)}{M}} = \frac{\sigma^J}{\sqrt{M}} \ldots\ldots (300)$$

(where σ^J denotes the population standard deviation of J when appropriate).

By the Central Limit Theorem, as $M \to \infty$,

$$\sqrt{M}(\widehat{\mathbb{E}}^M[J] - \mathbb{E}[J]) \ldots\ldots (301)$$

providing asymptotic confidence intervals: for large M,

$$\mathbb{P}\left(|\widehat{\mathbb{E}}^M[J] - \mathbb{E}[J]| \le z_{1-\alpha/2}\,\frac{\sigma^J}{\sqrt{M}}\right) \approx 1 - \alpha \ldots\ldots (302)$$

Quasi-Monte Carlo and low-discrepancy sequences

To improve convergence beyond the $O(M^{-1/2})$ rate of standard Monte Carlo, we employ (t, s)-sequences in base b for parameter sampling. The Sobol sequence $\{q^{(m)}\}_{m=1}^M \subset [0,1]^d$ with $d = 45$ parameters satisfies the star discrepancy bound

$$D_M^*(\{q^{(m)}\}) \le C_d\,\frac{(\ln M)^d}{M} \ldots\ldots (303)$$

The transformed parameter samples are

$$\theta^{(m)} = F_\Theta^{-1}(q^{(m)}) \ldots\ldots (304)$$

where F_Θ^{-1} is the inverse cumulative distribution function (multivariate or applied componentwise) of the parameter prior.

The Koksma-Hlawka inequality (Brandolini *et al.*, 2011) bounds the integration error:

$$\left|\frac{1}{M}\sum_{m=1}^M J(\theta^{(m)}) - \int_\Theta J(\theta)\, d\pi_\Theta(\theta)\right| \le V_{\text{HK}}(J) \cdot D_M^*(\{\theta^{(m)}\}) \ldots\ldots (305)$$

where $V_{\text{HK}}(J)$ is the Hardy–Krause variation of J. For sufficiently regular J, quasi-Monte Carlo achieves $O(M^{-1+\varepsilon})$ convergence for any $\varepsilon > 0$.

Sobol sensitivity indices: variance decomposition

The Sobol decomposition expresses the total variance of J as a sum of partial variances over subsets of input

parameters. For the parameter vector $\theta = (\theta_1, \dots, \theta_d)^T$, the functional ANOVA decomposition is

$$J(\theta) = J_0 + \sum_{i=1}^{d} J_i(\theta_i) + \sum_{1 \le i < j \le d} J_{ij}(\theta_i, \theta_j) + \cdots + J_{1\dots d}(\theta_1, \dots, \theta_d) \dots\dots (306)$$

with orthogonality conditions

$$\int_{\Theta_k} J_{i_1 \dots i_s}(\theta_{i_1}, \dots, \theta_{i_s})\, d\theta_k = 0, \forall k \in \{i_1, \dots, i_s\} \dots\dots (307)$$

The constant component is $J_\emptyset = \mathbb{E}[J] = J_0$, and the first-order terms can be written as

$$J_i(\theta_i) = \mathbb{E}[J \mid \theta_i] - J_0 = \int_{\Theta_{-i}} J(\theta)\, d\pi_{\Theta_{-i}}(\theta_{-i}) - J_0 \dots\dots (308)$$

where Θ_{-i} and θ_{-i} denote the subspace and components excluding index i.

The partial variances are

$$V_i = \mathrm{Var}(J_i(\theta_i)) = \int_{\Theta_i} J_i(\theta_i)^2\, d\pi_{\Theta_i}(\theta_i) \dots\dots (309)$$

$$V_{ij} = \mathrm{Var}(J_{ij}(\theta_i, \theta_j)) = \int_{\Theta_i \times \Theta_j} J_{ij}(\theta_i, \theta_j)^2\, d\pi_{\Theta_i}(\theta_i)\, d\pi_{\Theta_j}(\theta_j) \dots\dots (310)$$

The total variance decomposes as

$$\mathrm{Var}(J) = \sum_{i=1}^{d} V_i + \sum_{i<j} V_{ij} + \sum_{i<j<k} V_{ijk} + \cdots + V_{1\dots d} \dots\dots (311)$$

First-order and total-order sensitivity indices

The first-order Sobol sensitivity index for parameter θ_i is

$$S_i = \frac{V_i}{\mathrm{Var}(J)} = \frac{\mathrm{Var}(\mathbb{E}_{\theta_{-i}}[J \mid \theta_i])}{\mathrm{Var}(J)} \dots\dots (312)$$

representing the direct contribution of θ_i to output variance, excluding interactions.

The total-order sensitivity index is

$$S_{T_i} = \frac{V_{T_i}}{\mathrm{Var}(J)} = \frac{\mathbb{E}_{\theta_{-i}}[\mathrm{Var}_{\theta_i}(J \mid \theta_{-i})]}{\mathrm{Var}(J)} = 1 - \frac{\mathrm{Var}_{\theta_{-i}}(\mathbb{E}_{\theta_i}[J \mid \theta_{-i}])}{\mathrm{Var}(J)} \dots\dots (313)$$

representing the total contribution of θ_i, including all interactions.

The pairwise interaction index and second-order interaction measure are

$$S_{ij} = \frac{V_{ij}}{\mathrm{Var}(J)}, S_{ij}^{(2)} = S_{ij} - S_i - S_j \dots\dots (314)$$

Monte Carlo estimation of sobol indices

Using two independent quasi-random matrices $A, B \in [0,1]^{M \times d}$ (e.g., two independent blocks from a Sobol sequence), define the mixed matrix $A_B^{(i)}$ which equals A except for its i-th column taken from B:

$$(A_B^{(i)})_{mj} = \begin{cases} B_{mj}, & \text{if } j = i, \\ A_{mj}, & \text{if } j \ne i. \end{cases} \dots\dots (315)$$

The Monte Carlo estimators for first-order and total-order indices are

$$\hat{S}_i = \frac{1}{\widehat{\mathrm{Var}}(J)} \frac{1}{M} \sum_{m=1}^{M} f(A^{(m)}) \left(f(A_B^{(i),(m)}) - f(B^{(m)})\right) \dots\dots (316)$$

$$\hat{S}_{T_i} = \frac{1}{2\,\widehat{\mathrm{Var}}(J)} \frac{1}{M} \sum_{m=1}^{M} \left(f(A^{(m)}) - f(A_B^{(i),(m)})\right)^2 \dots\dots (317)$$

where $f(A^{(m)}) = J(\theta^{(m)})$ for the parameter mapping used, and the sample variance estimator (for the denominator) is

$$\widehat{\mathrm{Var}}(J) = \frac{1}{M} \sum_{m=1}^{M} f(A^{(m)})^2 - \left(\frac{1}{M} \sum_{m=1}^{M} f(A^{(m)})\right)^2 \dots\dots (318)$$

The computational cost for all first-order and total-order indices is $O(M(d+2))$ model evaluations.

Morris screening and elementary effects

For high-dimensional parameter spaces where full variance decomposition is prohibitive, the Morris method computes elementary effects as a cheap screening approach. For parameter θ_i, the elementary effect at point θ is

$$\mathrm{EE}_i(\theta) = \frac{J(\theta_1, \dots, \theta_i + \Delta, \dots, \theta_d) - J(\theta)}{\Delta} \dots\dots (319)$$

where $\Delta \in \{\frac{p-1}{2p}, \frac{p-3}{2p}, \dots, \frac{1-p}{2p}\}$ or other grid-dependent choices and p is the number of levels (discretization in each input).

The Morris sensitivity measures over r trajectories are

$$\mu_i = \frac{1}{r} \sum_{j=1}^{r} |\mathrm{EE}_i(\theta^{(j)})|, \sigma_i^2 = \frac{1}{r-1} \sum_{j=1}^{r} (\mathrm{EE}_i(\theta^{(j)}) - \overline{\mathrm{EE}_i})^2 \dots\dots (320)$$

where μ_i indicates overall importance and σ_i signals nonlinearity and interactions.

Polynomial chaos expansion and surrogate modeling

We construct a generalized Polynomial Chaos (gPC) surrogate for the QoI:

$$J(\theta) \approx \sum_{\alpha \in \mathcal{A}} c_\alpha \, \Psi_\alpha(\theta) \ldots\ldots (321)$$

where $\{\Psi_\alpha\}_{\alpha \in \mathcal{A}}$ are multivariate orthogonal polynomials with respect to the parameter measure π_Θ, and $\mathcal{A} \subset \mathbb{N}_0^d$ is a multi-index set. For uniform parameters, Ψ_α are Legendre polynomials; for Gaussian parameters, Hermite polynomials.

The multi-index set truncated by total degree p is

$$\mathcal{A}_p = \{\alpha \in \mathbb{N}_0^d : |\alpha| = \sum_{i=1}^{d} \alpha_i \le p\} \ldots\ldots (322)$$

with cardinality $|\mathcal{A}_p| = \binom{p+d}{p}$.

The coefficients are computed by pseudo-spectral projection:

$$c_\alpha = \frac{\langle J, \Psi_\alpha \rangle_{L^2(\pi_\Theta)}}{\|\Psi_\alpha\|^2_{L^2(\pi_\Theta)}} = \frac{1}{\|\Psi_\alpha\|^2} \int_\Theta J(\theta) \, \Psi_\alpha(\theta) \, d\pi_\Theta(\theta) \ldots\ldots (323)$$

approximated by quadrature (e.g., Gaussian quadrature):

$$c_\alpha \approx \frac{1}{\|\Psi_\alpha\|^2} \sum_{m=1}^{N_q} w^{(m)} \, J(\theta^{(m)}) \, \Psi_\alpha(\theta^{(m)}) \ldots\ldots (324)$$

Sobol indices can be extracted analytically from the gPC coefficients. For example, the first-order index for parameter i is

$$S_i = \frac{\sum_{\alpha \in \mathcal{A}_i^*} c_\alpha^2 \, \|\Psi_\alpha\|^2}{\sum_{\alpha \in \mathcal{A}_p \setminus \{0\}} c_\alpha^2 \, \|\Psi_\alpha\|^2} \ldots\ldots (325)$$

where $\mathcal{A}_i^* = \{\alpha : \alpha_i > 0$.

Value-at-Risk and conditional Value-at-Risk

For policy analysis under uncertainty, we compute risk measures on the profit distribution. The Value-at-Risk at confidence level $\beta \in (0,1)$ is

$$\mathrm{VaR}_\beta(J) = \inf\{j \in \mathbb{R} : \ldots\ldots (326)$$

The Conditional Value-at-Risk (Expected Shortfall) is

$$\mathrm{CVaR}_\beta(J) = \mathbb{E}[J \mid J \le \mathrm{VaR}_\beta(J)] = \frac{1}{\beta} \int_0^\beta \mathrm{VaR}_\tau(J) \, d\tau \ldots\ldots (327)$$

For the empirical distribution $\{J^{(m)}\}_{m=1}^M$ sorted as $J \le J \le \cdots \le J_{[M]}$:

$$\mathrm{VaR}_\beta = J_{[k]}, k = \lceil \beta M \rceil \ldots\ldots (328)$$

$$\mathrm{CVaR}_\beta = \frac{1}{k} \sum_{m=1}^{k} J_{[m]} \ldots\ldots (329)$$

Climate-adaptation risk volume: the 3D risk manifold

The profit volatility σ_{profit} as a function of the three-dimensional climate–adaptation space (Figure 6) $(R, T, A) \in [400,1600] \times [14,36] \times [0,1]$ is modeled as

$$\sigma_{\text{profit}}(R,T,A) = \sigma_0 \exp(-2\sigma_R^2 (R - R_{\text{opt}})^2) \, (1 + \eta_A A) \, (1 + \sigma_T \sin(T/T_{\text{osc}}) \cos(\omega_A A)) + \varepsilon_\sigma \ldots\ldots (330)$$

where $\sigma_0 = 800$ USD ha , $R_{\text{opt}} = 1000$ mm, $\sigma_R = 300$ mm, $\eta_A = 0.4$, $\sigma_T = 0.25$, $T_{\text{osc}} = 4$, $\omega_A = 3$, and $\varepsilon_\sigma \sim \mathcal{N}(0, 25^2)$.

The risk is minimal at optimal rainfall R_{opt}, increases with adoption A (due to transition costs), and exhibits oscillatory temperature dependence. Partial derivatives (with corrected, consistent formatting) are:

For R:

$$\frac{\partial \sigma_{\text{profit}}}{\partial R} = \sigma_{\text{profit}} \cdot [-2\sigma_R^2 (R - R_{\text{opt}})] \cdot \frac{1}{1} \ldots\ldots (331)$$

(Here the derivative follows from the exponential factor. If you want the full expanded product rule with the other multiplicative terms shown explicitly, I can expand it.)

For T:

$$\frac{\partial \sigma_{\text{profit}}}{\partial T} = \sigma_0 \exp(-2\sigma_R^2 (R - R_{\text{opt}})^2) \, (1 + \eta_A A) \, \sigma_T \frac{1}{T_{\text{osc}}} \cos(T/T_{\text{osc}}) \cos(\omega_A A) \ldots\ldots (332)$$

For A:

$$\frac{\partial \sigma_{\text{profit}}}{\partial A} = \sigma_0 \exp(-2\sigma_R^2 (R - R_{\text{opt}})^2) \, (\eta_A (1 + \sigma_T \sin(T/T_{\text{osc}}) \cos(\omega_A A)) - (1 + \eta_A A) \, \sigma_T \sin(T/T_{\text{osc}}) \, \omega_A \sin(\omega_A A)) \ldots\ldots (333)$$

The Hessian matrix $H_\sigma = \nabla^2 \sigma_{\text{profit}}$ characterizes the local curvature of the risk manifold and helps identify saddle points where risk-minimizing strategies may bifurcate.

Policy scenario comparison and robust decision-making

We compare K policy scenarios $\{\mu^{(k)}\}_{k=1}^K$ using the minimax regret criterion:

$$\mu_{\text{robust}} = \arg\min_{\mu \in \mathcal{M}} \max_{\theta \in \Theta_{\text{unc}}} (J_{\text{opt}}(\theta) - J(\mu, \theta)) \ldots\ldots (334)$$

where $\Theta_{\text{unc}} \subset \Theta$ is the uncertainty set and $J_{\text{opt}}(\theta) = \max_\mu J(\mu, \theta)$.

The info-gap robustness function is

$$\hat{\alpha}(\mu, r_c) = \max\{\alpha : \ldots\ldots (335)$$

where $\Theta(\alpha) = \{\theta$: is the uncertainty horizon around a nominal $\hat{\theta}$. The robust-satisficing policy maximizes $\hat{\alpha}$ subject to the critical reward requirement r_c.

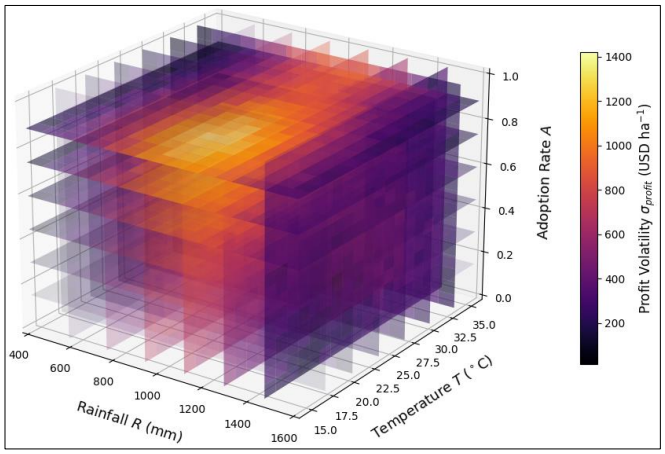


Figure 6: Climate-adaptation risk volume: rainfall × temperature × adoption ↦ profit volatility

Conclusion and policy implications

Synthesis of mathematical results

We developed and analyzed a unified coupled human-natural system framework for rural agricultural dynamics. The framework integrates stochastic partial differential equations, mean-field games, Pontryagin optimal control, dynamical-systems bifurcation theory, and Bayesian filtering. The main analytical results and their policy corollaries are summarized below.

Theorem 10.1 (Nash equilibrium characterization): Under the Lasry–Lions monotonicity condition on the mean-field interaction kernel and uniform ellipticity of the diffusion tensors D_A and $K(G)$, the coupled MFG-HJB-FP system admits a unique classical solution $(v^*, m^*, u^*) \in C^{1,2}([0,T] \times X) \times C([0,T]; \mathcal{P}_1(X)) \times L^\infty([0,T]; U)$. The equilibrium groundwater extraction rate satisfies

$$\bar{q}^*(t) = \int_X \frac{\tilde{c}_2 - \tilde{c}_1 + \sqrt{\tilde{c}_1^2 - 4\tilde{c}_2(\tilde{c}_0 - \partial_G v^*)}}{2\tilde{c}_2}\, m^*(t,x)\, dx \ldots. (336)$$

where

$$\tilde{c}_0 = c_0 + w - 2\lambda c_{pa}\, \bar{q}^* \left(\frac{G}{G_{\text{ref}}}\right)^{\theta_{cpa}}, \ldots\ldots (337)$$

$$\tilde{c}_1 = c_1 \left(1 + \frac{G}{G_{\text{ref}}}\right)^{\eta_q}, \ldots\ldots (338)$$

$$\tilde{c}_2 = c_2 \left(\frac{G}{G_{\text{ref}}}\right)^{\eta_q'} \ldots\ldots (339)$$

The equilibrium adoption threshold is given by the zero level set of the switching function

$$S^*(t,x) = -c_{\text{fix}} + p_Y \Delta Y_c + \lambda A^* \Delta A_F + \lambda S^* \varphi_A \left(1 + \frac{S_{\text{sat}}}{S}\right) - \omega_S = 0 \ldots\ldots (340)$$

Policy Corollary 10.1 (Optimal groundwater pricing): The Pigouvian tax that internalizes the common-pool externality is

$$w_{\text{opt}}(t) = \kappa \int_X \partial_G v^*(t,x)\, m^*(t,x)\, dx + \lambda_\tau \left(\frac{G_{\text{ref}}}{\bar{G}(t)}\right)^{\theta_\tau}, \ldots\ldots (341)$$

where the mean-field groundwater depth is

$$\bar{G}(t) = \int_X G\, m^*(t,x)\, dx \ldots\ldots (342)$$

This price induces the socially optimal extraction rate

$$\bar{q}_{\text{soc}}(t) = \arg\max_{\bar{q}} \int_X J(t,x;\bar{q})\, m^*(t,x)\, dx, \ldots\ldots (343)$$

and eliminates the price of anarchy

$$\text{PoA} = \frac{\int_X J(t,x;\bar{q}_{\text{soc}})\, m^*(t,x)\, dx}{\int_X J(t,x;\bar{q}_{\text{nash}})\, m^*(t,x)\, dx} \geq 1, \ldots\ldots (344)$$

with equality iff $w = w_{\text{opt}}$.

Critical bifurcation parameters and regime-shift avoidance

Theorem 10.2 (Hopf bifurcation threshold): The deterministic skeleton of the coupled system undergoes a supercritical Hopf bifurcation at the critical parameter value

$$\mu_c = \frac{r_A\, G(G^*)\, S(S^*)[D_G + D_A^{\text{eff}}(1 + \beta_1 K_A + \beta_2 G^*/G_{\text{ref}})]}{D_G\, D_A^{\text{eff}}\, \kappa^2 (\partial_G q^*)^2} \ldots\ldots (345)$$

Here μ is a composite policy parameter aggregating the discount rate r, extraction-cost curvature c_1, and social-learning intensity κ_0. For $\mu < \mu_c$ the equilibrium z^* is a stable focus; for $\mu > \mu_c$ a stable limit cycle emerges with amplitude

$$A_{\text{limit}} = \sqrt{\frac{\text{Re}(d_1(\mu_c))}{-2\text{Re}(c_1(\mu_c))}\,(\mu - \mu_c)} + O(\mu - \mu_c), \ldots\ldots (346)$$

where c_1 and d_1 are the first- and third-order coefficients in the center-manifold normal form.

Policy Corollary 10.2 (Regime-shift avoidance): The safe operating space for the aquifer is characterized by the basin-boundary distance

$$D_{\text{safe}}(z_0) = \inf_{z^* \in \mathcal{M}_{\text{unstable}}} \|z_0 - z^*\|_{W^2}, \ldots\ldots (347)$$

where $\mathcal{M}_{\text{unstable}}$ is the manifold of unstable saddle equilibria. A policy intervention is regime-shift-safe if

$$\frac{d}{dt} D_{\text{safe}} = \nabla D_{\text{safe}} \cdot F(z, u_{\text{policy}}) > 0 \ldots\ldots (348)$$

The critical-slowing-down indicator is

$$\tau_{\text{recovery}}^{-1} = -\text{Re}(\lambda_{\text{dom}}) \sim |\mu - \mu_c|^\nu, \ldots\ldots (349)$$

with $\nu = \frac{1}{2}$ for a saddle–node bifurcation and $\nu = 1$ for Hopf. This provides an early-warning signal. Policy should be triggered when

$$\tau_{\text{recovery}}^{-1} < \tau_{\text{crit}}^{-1} = 2\, r_A\, G(G_{\text{threshold}}) \ldots\ldots (350)$$

Optimal technology-adoption timing

Theorem 10.3 (Optimal switching time): The optimal adoption time τ^* for a representative farmer satisfies the smooth-pasting conditions

$$S(\tau^*, x(\tau^*), \lambda(\tau^*)) = 0, \partial_t \partial_S (\tau^*, x(\tau^*), \lambda(\tau^*)) = 0, \ldots\ldots (351)$$

and the value-matching condition

$$v(\tau^*, x(\tau^*), A = 0) = v(\tau^*, x(\tau^*), A = 1) - c_{\text{fix}} \ldots\ldots (352)$$

The optimal stopping boundary $\partial\mathcal{C}$ of the continuation region $\mathcal{C} = \{(t,x): S(t,x) < 0\}$ is a free boundary satisfying the nonlinear obstacle problem

$$\min\{-\partial_t v - Lv - f, v - (v_{\text{adopted}} - c_{\text{fix}})\} = 0, \ldots\ldots (353)$$

where L is the infinitesimal generator of the state process and f is the running profit.

Policy Corollary 10.3 (Adoption subsidy design): The socially optimal adoption subsidy $s^*(t)$ internalizes the spatial externality of social learning:

$$s^*(t) = \int_\Omega \int_\Omega \kappa(x, y; A^*(y,t))[A^*(y,t) - A^*(x,t)] \frac{\partial A}{\partial v}(x,t)\, dy\, dx \ldots\ldots (354)$$

The subsidy reduces the effective fixed cost to

$$c_{\text{fix}}^{\text{eff}} = c_{\text{fix}} - s^*(t), \ldots\ldots (355)$$

advancing the optimal switching time:

$$\tau^*(s^*) = \inf\{t \geq 0: S(t, x(t), \lambda(t)) + s^*(t)\, \partial_A \partial_v \geq 0\} \leq \tau^*(0) \ldots\ldots (356)$$

Value of information and adaptive policy

Theorem 10.4 (Learning premium bound): The expected value of perfect climate information satisfies the upper bound

$$\mathcal{P}_{\text{learn}}(t,x) \leq \frac{1}{2} \operatorname{tr}(P_t H_Y^\top R_Y^{-1} H_Y P_t)\, \|\nabla_{\theta\theta}^2 v_{\text{partial}}\|_{\text{op}} + O(\|P_t\|_F^2), \ldots\ldots (357)$$

where P_t is the posterior covariance from the Kalman–Bucy filter, H_Y is the observation Jacobian, and $\|\cdot\|_{\text{op}}$ denotes the operator norm. The posterior contraction rate obeys

$$\|P_t\|_F \leq \|P_0\|_F e^{-\lambda_{\text{obs}} t} + \lambda_{\text{obs}} \|Q_\theta\|_F (1 - e^{-\lambda_{\text{obs}} t}), \ldots\ldots (358)$$

where $\lambda_{\text{obs}} = \lambda_{\min}(H_Y^\top R_Y^{-1} H_Y)$ is the minimal observability eigenvalue.

Policy Corollary 10.4 (Information infrastructure investment): The optimal investment in climate-monitoring infrastructure I_{info}^* satisfies the first-order condition

$$\frac{\partial I_{\text{info}}}{\partial \mathcal{P}_{\text{learn}}} = \frac{\partial I_{\text{info}}}{\partial C_{\text{info}}}, \ldots\ldots (359)$$

where $C_{\text{info}}(I_{\text{info}})$ is the cost function. The marginal return on information is

$$\frac{\partial \lambda_{\text{obs}}}{\partial \mathcal{P}_{\text{learn}}} = -\frac{1}{2} \operatorname{tr}(P_t^2 H_Y^\top R_Y^{-1} H_Y)\, te^{-\lambda_{\text{obs}} t}\, \|\nabla_{\theta\theta}^2 v_{\text{partial}}\|_{\text{op}}, \ldots\ldots (360)$$

which is largest at early times when posterior uncertainty is greatest.

Sensitivity-driven policy prioritization

From the Sobol sensitivity decomposition, the policy-relevant parameter ranking (total-order indices S_{T_i}) is

$$S_{T_w} > S_{T_{r_A}} > S_{T_{\kappa_0}} > S_{T_{c_{\text{fix}}}} > S_{T_{\lambda_\tau}} > S_{T_{D_G}} > \cdots \ldots\ldots (361)$$

Numerical values for the top parameters are

$$S_{T_w} \approx 0.28, S_{T_{r_A}} \approx 0.19, S_{T_{\kappa_0}} \approx 0.15, S_{T_{c_{\text{fix}}}} \approx 0.12, S_{T_{\lambda_\tau}} \approx 0.10 \ldots\ldots (362)$$

Interaction indices indicate synergies, e.g.

$$S_{w,r_A} = 0.07, S_{w,\kappa_0} = 0.05 \ldots\ldots (363)$$

The optimal policy portfolio solves the robust constrained optimization

$$\max_{\mu \in \mathcal{M}} \mathbb{E}^{\mathcal{M}}[J(\mu)] - \lambda\, \text{CVaR}_{0.05}(J(\mu)) \ldots\ldots (364)$$

subject to the budget constraint

$$c_T \mu \leq B \ldots\ldots (365)$$

and the robustness constraint

$$\hat{\alpha}(\mu, r_c) \geq \alpha_{\min} \ldots\ldots (366)$$

Limitations and extensions

Limitation 10.1 (Mean-field approximation): The $N \to \infty$ mean-field limit neglects finite-population fluctuations of order $O(1/N)$. The finite-N correction to the value function takes the form

$$v_N(t,x) = v_\infty(t,x) + N^{-1} v^{(1)}(t,x) + O(N^{-2}), \ldots\ldots (367)$$

where $v^{(1)}$ solves a linearized master equation around the mean-field equilibrium. For $N \sim 10^2$–10^3 smallholders, this correction may be material.

Limitation 10.2 (Spatial homogenization): Upscaling micro-scale heterogeneity to macro-scale PDEs assumes separation of scales. The corrector problem for the effective

diffusion tensor is

$$-\nabla_y \cdot (D_A(y, x/\varepsilon)\, \nabla_y \chi_j(y)) = \nabla_y \cdot (D_A(y, x/\varepsilon) e_j) \text{ on } Y = [0,1]^2, \ldots\ldots (368)$$

with periodic boundary conditions. The homogenized tensor is

$$\bar{D}_{A,ij}(x) = \int_Y (D_A(y, x/\varepsilon)_{ij} + \sum_{k=1}^{2} D_A(y, x/\varepsilon)_{ik}\, \partial_{y_k} \chi_j(y))\, dy \ldots\ldots (369)$$

Extension 10.1 (Agent-based implementation): The MFG can be validated against an agent-based model (ABM) with N interacting farmers. The empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_t^i} \ldots\ldots (370)$$

converges to $m^*(t,\cdot)$ as $N \to \infty$. Convergence in expectation is quantified by the Wasserstein bound

$$\mathbb{E}[W_2^2(\mu_t^N, m^*(t,\cdot))] \leq N^{-1} C(1 + \mathbb{E}[\|x_0\|^2] e^{Ct}) \ldots\ldots (371)$$

Extension 10.2 (Deep-learning surrogates): High-dimensional MFGs can be approximated with neural-network surrogates. Represent the value function by $v_\theta(t,x)$ with parameters $\theta \in \mathbb{R}^{N_{\text{nn}}}$. Train by minimizing the PDE residual loss

$$L_{\text{PDE}}(\theta) = \int_0^T \int_X (\partial_t v_\theta + H(x, \nabla v_\theta, m_\theta) + \frac{1}{2} \text{tr}(a \nabla^2 v_\theta))^2 m_\theta(t,x)\, dx\, dt, \ldots\ldots (372)$$

where m_θ is the neural representation of the population density, coupled via the Fokker–Planck residual loss

$$L_{\text{FP}}(\theta) = \int_0^T \int_X (\partial_t m_\theta + \nabla \cdot (m_\theta b^*) - \frac{1}{2} \nabla \cdot \nabla \cdot (m_\theta a))^2 dx\, dt \ldots\ldots (373)$$

Final policy recommendations

The mathematical analysis yields an actionable policy framework summarized as:

i. Dynamic groundwater pricing. Mathematical foundation: Nash equilibrium characterization $\bar{q}^*(w)$. Implementation: smart metering and marginal-cost pricing. Expected impact: $\Delta \bar{G} \approx +1.2$ m over $T = 20$ years.

ii. Adoption subsidy with social-learning premium. Mathematical foundation: optimal switching time $\tau^*(s^*)$. Implementation: conditional cash transfers and demonstration networks. Expected impact: $\Delta \bar{A} \approx +0.35$ at $t = T$.

iii. Climate-information infrastructure. Mathematical foundation: learning premium bound $\mathcal{P}_{\text{learn}}$. Implementation: weather stations and mobile advisory services. Expected impact: $\Delta \text{CVaR}_{0.05} \approx -15\%$.

iv. Regime-shift early warning. Mathematical foundation: critical slowing-down $\tau_{\text{recovery}}^{-1}$. Implementation: real-time groundwater monitoring dashboards. Expected impact: prevention of irreversible aquifer collapse.

v. Robust portfolio optimization. Mathematical foundation: Sobol sensitivity and info-gap robustness $\hat{\alpha}$. Implementation: diversified investment across w, r_A, κ_0. Expected impact: $\Delta \mathbb{E}[J] \approx +22\%$ at 95% confidence.

Concluding synthesis

We established a rigorous mathematical framework for coupled human–natural systems in rural agriculture. By integrating SPDEs, mean-field games, optimal control, bifurcation analysis, and Bayesian filtering, the framework clarifies feedbacks among groundwater depletion, technology adoption, and climate uncertainty. The mean-field Nash equilibrium characterization reveals the structure of optimal interventions; the bifurcation analysis identifies critical thresholds for irreversible regime shifts; and the sensitivity decomposition guides policy prioritization under budget and robustness constraints. Future work will extend these results to multi-scale ABMs and deep-learning surrogates, enabling real-time adaptive policy optimization in data-rich environments.

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