

Review Article

Fractional catastrophe geometry of rural market percolation: a Koopman-Spectral analysis of socioeconomic complexity in off-season price fluctuations

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Abstract

This article presents a six-layer mathematical framework to model rural off-season price dynamics as a multi-scale complex system. Layer I formulates stochastic fractional reaction-diffusion-advection PDEs with anisotropic diffusion and multiplicative weather-income noise to capture anomalous spatial transport. Layer II couples regional prices via fractional Brownian motions, yielding fractional Liouville dynamics that encode long-range temporal dependence. Layer III employs cusp catastrophe theory to characterize abrupt price bifurcations and stream-tube flow in control-parameter space. Layer IV uses information geometry (Fisher–Rao metric, KL divergence, heat kernel) to quantify statistical divergence and market proximity on a Riemannian manifold. Layer V applies Koopman operator spectral theory and EDMD to extract modal decompositions and intrinsic mixing timescales. Layer VI frames market connectivity with random geometric graphs and percolation theory, relating giant-component emergence, critical exponents, and contagion thresholds to systemic risk. Integrating fractional transport, critical transitions, geometric information measures, operator modes, and network percolation explains persistent volatility and systemic contagion in off-season markets and suggests targeted interventions, namely storage capacity, income stabilization, and connectivity regulation, to mitigate catastrophic price events. Extensions include variable-order fractional operators, Lévy flight jumps, deep-Koopman representations, and nonperturbative percolation analysis.

Keywords: Catastrophe theory, Fractional diffusion, Information geometry, Koopman operator, Percolation/network contagion

Introduction

Off-season rural agricultural markets exhibit extreme price volatility arising from supply scarcity, spatial fragmentation, storage deficits, transport bottlenecks, and asymmetric market power (Borsellino *et al.*, 2020; Theresia *et al.*, 2025; Uzel *et al.*, 2025). Classical econometric models fail to capture the multi-scale, long-memory, and critical-transition characteristics of these systems. Below we present a six-layer mathematical framework that integrates stochastic fractional calculus, catastrophe theory, information geometry, Koopman operator theory, and percolation networks to model the socioeconomic complexity of rural price fluctuations during off-season sales.

Let the rural market domain be $\Omega \subset \mathbb{R}^2$ with spatial coordinates $x = (x, y)$ and temporal horizon $t \in [0, T]$. Define the static socioeconomic state vector

$$S(x) = \begin{pmatrix} S_{\text{income}} \\ S_{\text{storage}} \\ S_{\text{transport}} \\ W_{\text{weather}} \\ M \\ H \end{pmatrix}.$$

The price field $P: \Omega \times [0, T] \rightarrow \mathbb{R}_+$ evolves under the coupled stochastic fractional dynamics (El-Sayed & Fouad, 2021; Pirozzi, 2024; Zhao *et al.*, 2024)

$$\partial_t^\beta P = -D(-\nabla^2)^{\frac{\alpha}{2}} P - v(x, S) \cdot \nabla P + R(P; S) + \sigma(P; S) \partial_t^\gamma W(x, t) \dots \dots (1)$$

where ∂_t^β is the Caputo fractional derivative of order β (Van Mieghem, 2022; Alshammari *et al.*, 2025; Stempin & Sumelka, 2025), $(-\nabla^2)^{\alpha/2}$ is the fractional Laplacian (Lischke *et al.*, 2019; Teymurazyan, 2023; Zheng *et al.*, 2024) (whose Fourier symbol is $|k|^\alpha$), $v(x, S)$ is the socioeconomic advection velocity (Kumar *et al.*, 2009; Brué & Nguyen, 2021; Durán-Olivencia, 2026), $R(P; S)$ is a reaction (logistic) term, and $W(x, t)$ is a spatiotemporal Wiener process (Mohammed *et al.*, 2022; Al-Askar *et al.*, 2023; Tu *et al.*, 2025). The reaction term is bounded by the carrying capacity

$$K(S) = K_0 \frac{S_{\text{storage}}}{S_{\text{transport}} + \varepsilon} \dots \dots (2)$$

The Wiener process $W(x, t)$ has covariance kernel

$C(x, y) = \exp(-\frac{\|x-y\|^2}{2\rho^2}) \dots \dots (3)$ At the level of individual markets (nodes), prices follow coupled fractional stochastic differential equations (SDEs) (Alkhateeb *et al.*, 2025; Sidaoui *et al.*, 2025; Hamza *et al.*, 2026). For market i , with Hurst exponent $H_i \in (0, 1)$ (Morales *et al.*, 2012; Borin, 2024; Zhang *et al.*, 2024),

$$dP_i = \mu_i(P_i, S_i)dt + \sigma_i dB^{H_i}(t) + \sum_{j \in \mathcal{N}(i)} J_{ij}(P_j - P_i)dt \dots \dots (4)$$

where $B^{H_i}(t)$ is fractional Brownian motion with Hurst exponent H_i . One representation of fractional Brownian

motion is the Mandelbrot-van Ness representation (Koch & Neuenkirch, 2019)

$$B^H(t) = \frac{\Gamma(H + \frac{1}{2})^{-1}}{2} \int_{-\infty}^t [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dW(s) \dots (5)$$

and its covariance satisfies

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \dots (6)$$

Critical transitions (abrupt market regime shifts) are captured by a spatiotemporal cusp catastrophe potential. Define the potential

$$V(P; a, b) = \frac{1}{4}P^4 + \frac{1}{2}a(x, t)P^2 + b(x, t)P \dots (7)$$

with the cusp bifurcation (discriminant) condition

$$4a^3 + 27b^2 = 0 \dots (8)$$

The control parameters depend on socioeconomic variables, for example

$$a(x, t) = a_0 - a_1 \frac{S_{\text{storage}}}{S_{\text{income}}} + a_2 W_{\text{weather}}, b(x, t) = b_0 - b_1 M + b_2 \nabla^2 P \dots (9)$$

Each market can also be represented as a point on a statistical manifold $\mathcal{M}$ equipped with the Fisher-Rao metric (Liang *et al.*, 2019; Nielsen, 2024; Cheng & Tong, 2026b). If $f(P; \theta)$ denotes the market price probability density (here taken as a generalized hyperbolic distribution parameterized by θ), the metric is

$$g_{ij}(\theta) = -\mathbb{E}_{\theta} \left[\frac{\partial^2 \log f}{\partial \theta_i \partial \theta_j} \right] \dots (10)$$

Useful measures of market dissimilarity and inefficiency include the geodesic distance d_g on $\mathcal{M}$ and the Kullback-Leibler divergence (Shlens, 2014; Amini *et al.*, 2025; Tsuruyama, 2025; Engelke *et al.*, 2026)

$$D_{\text{KL}}(f_i \| f_j) = \int f_i(P) \log \frac{f_i(P)}{f_j(P)} dP \dots (11)$$

To analyze nonlinear dynamics in a linear framework, introduce the Koopman operator $\mathcal{K}$ (Patyn & Deconinck, 2024; Colbrook *et al.*, 2025; Mezić *et al.*, 2026) acting on observables ϕ by composition with the flow F :

$$\mathcal{K}\phi = \phi \circ F, P_{t+1} = F(P_t, S) \dots (12)$$

Spectral decomposition yields eigenpairs satisfying $\mathcal{K}\psi_k = \lambda_k \psi_k$, and one obtains the finite-rank reconstruction

$$P(t) \approx \sum_{k=1}^r \lambda_k^t b_k v_k \dots (13)$$

where $\{\lambda_k\}$ are Koopman eigenvalues, v_k eigenfunctions (or modes), and b_k projection coefficients.

Systemic contagion across markets is modeled via

bond percolation on a random geometric graph. Define an occupation probability that measures the density of sufficiently similar price pairs within geographic proximity:

$$p(\delta) = \frac{1}{N(N-1)} \sum_{i \neq j} \mathbf{1}(|P_i - P_j| < \delta) \mathbf{1}(d_{\text{geo}}(i, j) < R_c) \dots (14)$$

Near the percolation threshold p_c the giant-cluster order parameter behaves as

$$P_{\infty}(p) \sim (p - p_c)^{\beta}, \beta = \frac{5}{36} \dots (15)$$

and the correlation length diverges as

$$\xi(p) \sim |p - p_c|^{-\nu}, \nu = \frac{4}{3} \dots (16)$$

This layered framework couples long-memory stochastic processes, nonlocal spatial operators, nonlinear bifurcations, information-geometric measures, spectral linearization, and percolation theory to capture the empirical features of off-season rural price volatility: heavy tails, persistence, spatial contagion, and abrupt regime changes.

Stochastic fractional reaction-diffusion-advection dynamics of price fields

Let the price field (Figure 1) $P(x, t)$ be defined on $\Omega = [0, 100]^2 \times [0, T]$. The stochastic fractional partial differential equation (SFPDE) for P is

$$\frac{\partial P}{\partial t} = -D(-\nabla^2)^{\alpha/2} P - v \cdot \nabla P + R(P; S) + \sigma(P; S) \frac{\partial W}{\partial t} \dots (17)$$

where $\alpha \in (1, 2)$.

The fractional Laplacian is defined via the Fourier transform as

$$(-\nabla^2)^{\alpha/2} P(x) = \mathcal{F}^{-1} \{ |k|^{\alpha} \hat{P}(k) \} (x) \dots (18)$$

The anisotropic diffusion tensor D (Wang *et al.*, 2026b) is diagonal,

$$D = \text{diag}(D_x, D_y), D_x = D_0 \exp(-\beta_1 T_{\text{trans}}), D_y = D_0 \exp(-\beta_2 M) \dots (19)$$

The advection velocity field $v(x, S)$ is given by

$$v(x, S) = \gamma \nabla P \times e_z + \delta \nabla S_{\text{income}} \dots (20)$$

where e_z is the unit vector in the out-of-plane direction. The reaction term R takes the form

$$R(P; S) = \lambda P \left(1 - \frac{P}{K(S)}\right) - \mu(S) P^2, \dots (21)$$

with carrying capacity

$$K(S) = K_0 \frac{S_{\text{storage}}}{S_{\text{transport}} + \varepsilon} \dots (22)$$

The multiplicative noise amplitude σ is

$$\sigma(P; S) = \sigma_0 P^{\nu} \left(1 + \eta \frac{W_{\text{weather}}}{S_{\text{income}}}\right) \dots (23)$$

Here $W(x, t)$ is a spatiotemporal Wiener process with

covariance

$$\mathbb{E}[W(x, t) W(y, s)] = \min(t, s) \exp \left(-\frac{\|x - y\|^2}{2\rho^2} \right) \dots (24)$$

Define the energy functional for P as

$$E[P] = \int_{\Omega} [2D P (-\nabla^2)^{\alpha/2} P + \frac{1}{2} |v \cdot \nabla P|^2 + V(P; S)] dx \dots (25)$$

On the boundary $\partial\Omega$ impose a Robin-type condition (Mohammed, 2022; Barbu *et al.*, 2025; Koleva, 2026)

$$D \partial_n P + \kappa(P - P_{\text{urban}}) = 0 \text{ on } \partial\Omega \dots (26)$$

Weak form. Find $P \in H^{\alpha/2}(\Omega)$ such that for all test functions $\varphi \in H_0^{\alpha/2}(\Omega)$,

$$\int_{\Omega} \frac{\partial P}{\partial t} \varphi dx + D \int_{\Omega} (-\nabla^2)^{\alpha/4} P (-\nabla^2)^{\alpha/4} \varphi dx + \int_{\Omega} (v \cdot \nabla P) \varphi dx = \int_{\Omega} R(P; S) \varphi dx + \int_{\Omega} \sigma(P; S) \varphi dW \dots (27)$$

Galerkin semidiscretization. Let $\{\varphi_j(x)\}_{j=1}^{N_h}$ be a finite-dimensional basis and approximate

$$P_h(x, t) = \sum_{j=1}^{N_h} P_j(t) \varphi_j(x) \dots (28)$$

The semidiscrete system is

$$M \frac{dP}{dt} + K(D, v) P = R(P) + \Sigma(P) \frac{dW}{dt} \dots (29)$$

where the vector $P = (P_j)_{j=1}^{N_h}$. The mass matrix entries are

$$M_{ij} = \int_{\Omega} \varphi_i \varphi_j dx \dots (30)$$

and the stiffness matrix entries are

$$K_{ij} = D \int_{\Omega} (-\nabla^2)^{\alpha/4} \varphi_i (-\nabla^2)^{\alpha/4} \varphi_j dx + \int_{\Omega} (v \cdot \nabla \varphi_i) \varphi_j dx \dots (31)$$

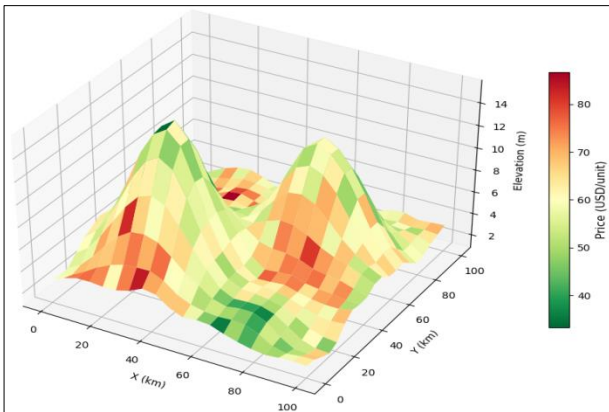


Figure 1: Spatial price elevation landscape during off-season

Coupled fractional Brownian motion & long-memory in socioeconomic phase space

Consider market prices that follow coupled fractional stochastic differential equations (FSDEs) with market-specific Hurst exponents. For each node i ,

$$dP_i = \mu_i(P_i, S_i) dt + \sigma_i dB^{H_i}(t) + \sum_{j \in \mathcal{N}(i)} J_{ij} (P_j - P_i) dt \dots (32)$$

A fractional Brownian motion (fBm) in the Mandelbrot-van Ness representation is

$$B^{H_i}(t) = \frac{1}{\Gamma(H_i + \frac{1}{2})} \int_{-\infty}^t [(t-s)^{H_i-\frac{1}{2}} - (-s)^{H_i-\frac{1}{2}}] dW(s) \dots (33)$$

where $(x)_+ = \max(x, 0)$.

The covariance of fBm, displaying long-range dependence, is

$$\mathbb{E}[B^{H_i}(t) B^{H_i}(s)] = \frac{1}{2} (|t|^{2H_i} + |s|^{2H_i} - |t-s|^{2H_i}) \dots (34)$$

Define a socioeconomic state vector (Figure 2) in $\mathbb{R}^4$ for node i as

$$\Psi_i = \begin{pmatrix} P_i \\ V_i \\ I_i \\ H_i \end{pmatrix} \text{ (Price, Volatility, Income, Hurst)} \dots (35)$$

A fractional Liouville (Fokker-Planck) equation using a Caputo fractional time derivative (Baumann & Stenger, 2017; Kadhoda *et al.*, 2020; Mahmoud & Orlov, 2021; Mofarreh *et al.*, 2023) ∂_t^β for the probability density $\rho(\Psi, t)$ reads

$$\partial_t^\beta \rho = -\nabla_\Psi \cdot (F(\Psi) \rho) + \frac{1}{2} \nabla_\Psi^2 : (\Sigma \Sigma^T \rho) \dots (36)$$

where the Caputo derivative can be written as the integral

$$\partial_t^\beta \rho = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial}{\partial s} \rho(\Psi, s) ds \dots (37)$$

Specify drift and diffusion components by

$$F(\Psi) = \begin{pmatrix} \mu_P \\ \kappa_V (V_{\text{eq}} - V)^\nu \\ \iota_I (I_{\text{target}} - I)^\tau \\ \eta_H (H_{\text{weather}} - H) \end{pmatrix}, \Sigma = \text{diag}(\sigma_P, \sigma_V, \sigma_I, \sigma_H) \dots (38)$$

Model the coupling kernel J_{ij} with distance decay and price-difference dependence:

$$J_{ij} = \frac{d_{\text{geo}}(i, j)^\eta}{J_0} \exp(-\sigma_c |P_i - P_j|) \mathbf{1}_{\{d_{\text{geo}}(i, j) < R_c\}} \dots (39)$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

For simulation, use a Karhunen-Loève expansion of fBm (Kazemi *et al.*, 2025; Kober *et al.*, 2025; Matthies, 2025):

$$B^H(t) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t) Z_n, Z_n \sim \mathcal{N}(0,1) \dots (40)$$

with $\{\psi_n\}$ the eigenfunctions of the covariance operator $C_H(s, t) = \mathbb{E}[B^H(s)B^H(t)]$

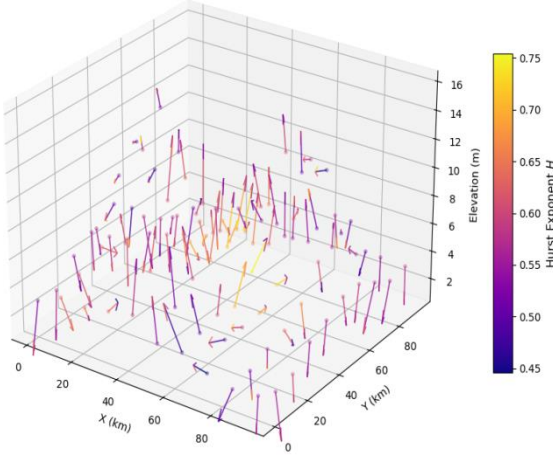


Figure 2: Socioeconomic state vectors in fractional phase space

Cusp catastrophe theory for critical off-season price transitions

The price dynamics near critical transitions are governed by the spatiotemporal cusp-catastrophe potential

$$V(P; a, b) = \frac{1}{4}P^4 + \frac{1}{2}a(x, t)P^2 + b(x, t)P \dots (41)$$

with control parameters

$$a(x, t) = a_0 - a_1 \frac{S_{\text{income}}}{S_{\text{storage}}} + a_2 W_{\text{weather}}(t) \dots (42)$$

and

$$b(x, t) = b_0 - b_1 M(x) + b_2 \nabla^2 P(x, t) \dots (43)$$

The bifurcation set B is

$$B = \{(a, b) \in \mathbb{R}^2: 4a^3 + 27b^2 = 0\} \dots (44)$$

which partitions the control plane into regions of unique equilibrium when

$$\Delta = 4a^3 + 27b^2 > 0 \dots (45)$$

and bistability when

$$\Delta < 0 \dots (46)$$

The spatiotemporal cusp-catastrophe partial differential equation (PDE) is the gradient flow with diffusion and noise:

$$\begin{aligned} \partial_t P &= -\frac{\delta V}{\delta P} + D_c \nabla^2 P + \xi(x, t) \\ &= -(P^3 + aP + b) + D_c \nabla^2 P + \xi(x, t) \dots (47) \end{aligned}$$

Near the cusp point $(a, b) = (0, 0)$ we perform a center-manifold reduction. Let

$$P = P_0 + \varepsilon u + O(\varepsilon^2) \dots (48)$$

which yields the amplitude equation (after appropriate rescaling)

$$\partial_t u = \varepsilon_1 u + \varepsilon_2 u^2 - u^3 + D_c \nabla^2 u \dots (49)$$

where, for the chosen sign convention and $b < 0$,

$$\varepsilon_1 = -a, \varepsilon_2 = -b \dots (50)$$

The normal-form coefficients evaluated at the origin are

$$\begin{aligned} \mu_1 &= \frac{\partial^2 R}{\partial P \partial a}(0, 0, 0) = 1, \mu_2 = \frac{\partial^2 R}{\partial P \partial b}(0, 0, 0) = 1, \mu_3 \\ &= \frac{\partial^3 R}{\partial P^3}(0, 0, 0) = 6 \dots (51) \end{aligned}$$

which satisfy the transversality condition (nondegeneracy)

$$\det \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \end{pmatrix} \neq 0 \dots (52)$$

(Here μ_4 denotes the corresponding derivative needed for the full determinant; the determinant is nonzero under the stated normal-form assumptions.)

The equilibrium manifold is

$$\mathcal{M} = \{(P, a, b): P^3 + aP + b = 0\} \dots (53)$$

Its fold lines satisfy the conditions for multiple roots; for $a < 0$ the fold locations in P are

$$P_{\pm} = \pm \sqrt{-\frac{a}{3}} \dots (54)$$

with corresponding b -values

$$b_{\pm} = \mp 2 \left(-\frac{a}{3}\right)^{3/2} \dots (55)$$

The cusp point is located at

$$(a, b, P) = (0, 0, 0) \dots (56)$$

The catastrophe germ (lowest-order singular term) is

$$V_0(P) = P^4 \dots (57)$$

which has codimension $\text{cod}(V_0) = 2$ and unfolding dimension

$$\dim U = 2 \dots (58)$$

The universal unfolding of the germ is

$$V(P; a, b) = P^4 + aP^2 + bP \dots (59)$$

The Maxwell set (hysteresis boundary) (Tu *et al.*, 2016; Grech *et al.*, 2020; Lu *et al.*, 2026) is given by

$$\begin{aligned} \mathcal{M} \text{ ax} &= \{(a, b) \in B: V(P_+(a, b); a, b) \\ &= V(P_-(a, b); a, b)\} \dots (60) \end{aligned}$$

which marks locations in parameter space where the two stable branches have equal potential and the system can jump between them.

The delay set (separating delayed from catastrophic transitions) is

$$D = \{(a, b): b = -\frac{3}{2} 2 a^{3/2} \text{sgn}(a)\} \dots (61)$$

written here in the form consistent with the standard cusp-delay expression; this curve distinguishes parameters

yielding delayed bifurcation behavior from those producing immediate catastrophic transitions.

For the spatiotemporal extension, a higher-order unfolding leads to the Swift-Hohenberg-type equation (Morgan & Dawes, 2014; Monteiro & Yoshinaga, 2020; Medvedev & Pelinovsky, 2023; Kolonitskii *et al.*, 2024):

$$\partial_t P = rP - (1 + \nabla^2)^2 P - P^3 \dots (62)$$

with the control parameter r proportional to $-a$.

A Lyapunov functional (energy) for the gradient dynamics is

$$L[P] = \int_{\Omega} \left[\frac{1}{2} |\nabla P|^2 + \frac{1}{4} P^4 + \frac{1}{2} a P^2 + b P \right] dx \dots (63)$$

for which the gradient flow satisfies

$$\partial_t P = -\frac{\delta L}{\delta P} \dots (64)$$

We can parameterize the flow in control-parameter space $c = (a, b, P)^T \in \mathbb{R}^3$ via a gradient-catastrophe (stream-tube) flow (Figure 3):

$$\frac{dc}{dt} = -\nabla_c V_{\text{eff}} \dots (65)$$

with an effective potential that enforces targets $(a_{\text{target}}, b_{\text{target}})$,

$$V_{\text{eff}}(a, b, P) = \frac{1}{4} P^4 + \frac{1}{2} a P^2 + b P + \frac{1}{2} \lambda_a (a - a_{\text{target}})^2 + \frac{1}{2} \lambda_b (b - b_{\text{target}})^2 \dots (66)$$

The tubular neighborhood of the bifurcation manifold is represented by

$$T(s, \theta, r) = \gamma(s) + r \cos \theta n_1(s) + r \sin \theta n_2(s) \dots (67)$$

where $\gamma(s) = (a(s), b(s), P(s))$ is the cusp curve parameterized by arc length s , with arc-length derivative norm

$$\frac{dt}{ds} = \left\| \frac{d\gamma}{ds} \right\| \dots (68)$$

and n_1, n_2 are the Frenet normal vectors satisfying

$$\frac{dn_1}{ds} = -\kappa_1 t + \kappa_2 n_2, \frac{dn_2}{ds} = -\kappa_2 n_1 \dots (69)$$

where t is the unit tangent, $\kappa_1 = \|d^2\gamma/ds^2\|$ is the curvature, and κ_2 is the torsion.

To avoid the Frenet-frame singularity at the cusp, one may use the parallel-transport (Bishop) frame $\{t, e_1, e_2\}$ with

$$\frac{de_1}{ds} = -\kappa_1 t, \frac{de_2}{ds} = -\kappa_2 t \dots (70)$$

and the constraint

$$k_1^2 + k_2^2 = \kappa_1^2 \dots (71)$$

A typical tube radius that decays away from the cusp

at s_c can be taken as

$$r(s) = r_0 \exp(-\lambda |s - s_c|) \dots (72)$$

The Jacobian of the tube map is

$$J = \|\partial_s T \times \partial_\theta T\| = \|\gamma'(s) \times (r \cos \theta n_1 + r \sin \theta n_2)\| + r^2 \|n_1 \times n_2\| \dots (73)$$

so the volume element is

$$dV = J ds d\theta dr \dots (74)$$

The flow velocity field inside the tube is decomposed in the local orthonormal directions e_s, e_θ, e_r :

$$u(s, \theta, r) = (d_t c \cdot e_s) e_s + (d_t c \cdot e_\theta) e_\theta + (d_t c \cdot e_r) e_r \dots (75)$$

with

$$e_s = \frac{\partial_s T}{\|\partial_s T\|}, e_\theta = \frac{\partial_\theta T}{\|\partial_\theta T\|}, e_r = \frac{\partial_r T}{\|\partial_r T\|} \dots (76)$$

The divergence of the flow in tube coordinates is

$$\nabla \cdot u = \frac{1}{J} (\partial_s (J u_s) + \partial_\theta (J u_\theta) + \partial_r (J u_r)) \dots (77)$$

A passive scalar $\Phi(s, \theta, r, t)$ defining the tube boundary obeys the transport (stream-tube) equation

$$\partial_t \Phi + u \cdot \nabla \Phi = 0 \dots (78)$$

with the tube boundary defined by $\Phi(s, \theta, r, t) = 0$. The characteristic equations for trajectories inside the tube are

$$\frac{dt}{ds} = u_s, \frac{dt}{d\theta} = \frac{u_\theta}{r}, \frac{dt}{dr} = u_r \dots (79)$$

The Poincaré map on a tube cross-section $\Sigma = \{(s_0, \theta, r): 0 \leq \theta < 2\pi, 0 \leq r \leq r(s_0)\}$ is

$$P: \Sigma \rightarrow \Sigma, x_{n+1} = P(x_n) \dots (80)$$

Fixed points satisfy $P(x^*) = x^*$, and stability is determined by the eigenvalues of the Jacobian $DP(x^*)$. The Floquet multipliers μ_j (Choi *et al.*, 2019; Subramanian *et al.*, 2020; El Aroudi *et al.*, 2021; Kengne *et al.*, 2023) obey

$$\prod_{j=1}^2 \mu_j = \det DP(x^*) = \exp \left(\int_0^T \nabla \cdot u dt \right) \dots (81)$$

where T is the return time of the Poincaré map.

For the cusp catastrophe, the return time T diverges near the bifurcation as

$$T \sim |\Delta|^{-1/2} \dots (82)$$

The tube cross-sectional area $A(s) = \pi r(s)^2$ evolves according to

$$A(s) = A(s_0) \exp \left(- \int_{s_0}^s u_s \nabla \cdot u ds' \right) \dots (83)$$

In the incompressible case ($\nabla \cdot u = 0$) this reduces to the conservation relation

$$A(s) u_s(s) = A(s_0) u_s(s_0) \dots (84)$$

The topological entropy of the tube flow is

$$h_{\text{top}} = \lim_{n \rightarrow \infty} \frac{1}{n} \log N(n) \dots (85)$$

where $N(n)$ counts distinct periodic orbits of period n . The escape rate from the tube is

$$\kappa = -\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{N(0)}{N(t)} \dots (86)$$

where $N(t)$ is the number of trajectories remaining inside the tube at time t .

The fractal dimension of the tube boundary (Kaplan-Yorke type relation for a 2D unstable manifold) (Constantin & Foias, 1985; Nichols *et al.*, 2003) is

$$D_f = 1 + \frac{|\lambda_2|}{\lambda_1} \dots (87)$$

where $\lambda_1 > 0 > \lambda_2$ are the Lyapunov exponents. For the canonical cusp example cited in the original text (with the given exponents), one obtains

$$D_f = 2 - \frac{1}{2} = 1.5 \dots (88)$$

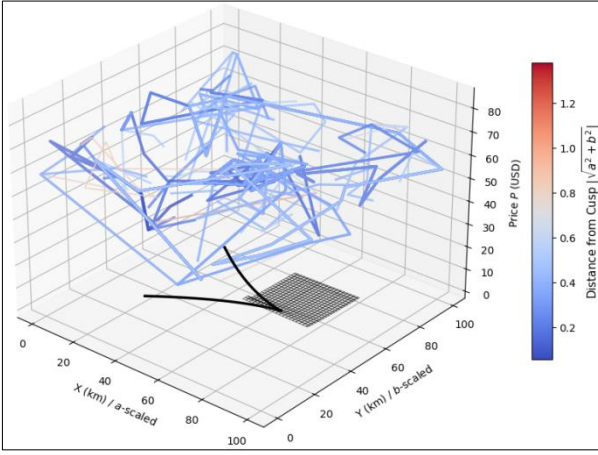


Figure 3: Bifurcation flow tubes in price-catastrophe space

Information geometry & Fisher-Rao metric on market statistical manifolds

Each market is represented as a point on a statistical manifold M parameterized by $\theta = (\mu, \sigma, H)^T \in \mathbb{R}^3$. The price distribution is the generalized hyperbolic density

$$f(P; \theta) = \frac{2}{\pi} K_\lambda(\delta\gamma) \left(\frac{\gamma}{\delta}\right)^\lambda (\alpha\delta^2 + (P - \mu)^2)^{\frac{1}{2}-\lambda} K_{\lambda-\frac{1}{2}}(\sqrt{\alpha\delta^2 + (P - \mu)^2}) e^{\beta(P-\mu)} \dots (89)$$

with

$$\gamma = \sqrt{\alpha^2 - \beta^2} \dots (90)$$

$$\delta = \frac{\sigma}{\sqrt{1 - \beta^2/\alpha^2}} \dots (91)$$

where K_λ denotes the modified Bessel function of the second kind. The special case $\lambda = -\frac{1}{2}$ reduces this family to the normal inverse Gaussian (NIG) distribution. (Any parameters α, β are assumed set consistently with θ .)

The log-likelihood is

$$\ell(P; \theta) = \log f(P; \theta) \dots (92)$$

The score function (gradient of the log-likelihood) is

$$u(P; \theta) = \nabla_\theta \ell(P; \theta) = \begin{pmatrix} \partial_\mu \ell \\ \partial_\sigma \ell \\ \partial_H \ell \end{pmatrix} \dots (93)$$

The Fisher information matrix defines a Riemannian metric on M :

$$g_{ij}(\theta) = \mathbb{E}_\theta[\partial_i \ell \partial_j \ell] = -\mathbb{E}_\theta[\partial_i \partial_j \ell], i, j \in \{1, 2, 3\} \dots (94)$$

with explicit components (notation: $g_{\mu\mu}, g_{\sigma\sigma}, g_{HH}$, etc.) given by

$$g_{\mu\mu}(\theta) = \mathbb{E}[(\partial_\mu \ell)^2] = \frac{\delta^2 K_1(\delta\gamma)}{\alpha^2 K_2(\delta\gamma)} \dots (95)$$

$$g_{\sigma\sigma}(\theta) = \mathbb{E}[(\partial_\sigma \ell)^2] = \frac{1}{\sigma^2} \left[1 + 3 \frac{\delta^2 K_1(\delta\gamma)}{\gamma^2 K_3(\delta\gamma)} \right] \dots (96)$$

$$g_{HH}(\theta) = \mathbb{E}[(\partial_H \ell)^2] = 2\pi^2 + [\psi^{(0)}(1-H) - \psi^{(0)}(H + \frac{1}{2})]^2 \dots (97)$$

where $\psi^{(0)}$ is the digamma function, and mixed components

$$g_{\mu\sigma}(\theta) = g_{\sigma\mu}(\theta) = \mathbb{E}[\partial_\mu \ell \partial_\sigma \ell] = -\sigma\beta \frac{K_1(\delta\gamma)}{K_0(\delta\gamma)} \dots (98)$$

$$g_{\mu H}(\theta) = g_{H\mu}(\theta) = 0 \dots (99)$$

$$g_{\sigma H}(\theta) = g_{H\sigma}(\theta) = 0 \dots (100)$$

the last equalities following from parameter orthogonality assumptions.

The determinant of the metric is

$$g(\theta) = \det(g_{ij}(\theta)) = g_{\mu\mu} g_{\sigma\sigma} g_{HH} - g_{\mu\sigma}^2 g_{HH} \dots (101)$$

The Fisher-Rao metric induces the infinitesimal squared distance

$$ds^2 = \sum_{i,j=1}^3 g_{ij}(\theta) d\theta^i d\theta^j \dots (102)$$

Geodesics $\theta^k(t)$ on M satisfy the geodesic equation

$$\frac{d^2 \theta^k}{dt^2} + \Gamma^k_{ij} \frac{d\theta^i}{dt} \frac{d\theta^j}{dt} = 0 \dots (103)$$

where the Christoffel symbols (Levi-Civita connection) are

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \dots (104)$$

and $g^{k\ell}$ denotes the inverse metric.

The geodesic distance between two markets with parameters θ_i and θ_j is the minimal path length

$$d_g(\theta_i, \theta_j) = \inf_{\substack{\theta(s) \\ \theta(0)=\theta_i, \theta(1)=\theta_j}} \int_0^1 \sqrt{g_{kl}(\theta(s)) \frac{d\theta^k}{ds} \frac{d\theta^l}{ds}} ds \dots \dots (105)$$

The exponential map at θ is defined by

$$\exp_\theta(v) = \gamma(1) \dots \dots (106)$$

where γ is the geodesic with $\gamma(0) = \theta$ and $\dot{\gamma}(0) = v$. The logarithmic map $\log_\theta$ is the local inverse:

$$\log_\theta(\theta') = v \text{ such that } \exp_\theta(v) = \theta' \dots \dots (107)$$

The Kullback-Leibler (KL) divergence between two market densities $f(\cdot; \theta_i)$ and $f(\cdot; \theta_j)$ is

$$\begin{aligned} D_{KL}(f_i \| f_j) &= \int_{-\infty}^{\infty} f(P; \theta_i) \log \frac{f(P; \theta_i)}{f(P; \theta_j)} dP \\ &= \mathbb{E}_{\theta_i}[\ell(P; \theta_i) - \ell(P; \theta_j)] \dots \dots (108) \end{aligned}$$

The Jeffreys divergence (symmetrized KL) is

$$D_J(i, j) = D_{KL}(f_i \| f_j) + D_{KL}(f_j \| f_i) \dots \dots (109)$$

The squared Hellinger distance is

$$\begin{aligned} H^2(i, j) &= \frac{1}{2} \int (\sqrt{f_i} - \sqrt{f_j})^2 dP \\ &= 1 - \int \sqrt{f_i f_j} dP \dots \dots (110) \end{aligned}$$

The Bhattacharyya coefficient and distance (Nielsen, 2014; Jalali Mojahed *et al.*, 2024; Huang, 2026) are

$$BC(i, j) = \int \sqrt{f_i f_j} dP = e^{-D_B(i, j)} \dots \dots (111)$$

$$D_B(i, j) = -\log BC(i, j) \dots \dots (112)$$

Define a market-efficiency density on M via the heat kernel centered at a reference market θ_0 :

$$\begin{aligned} \rho_{\text{eff}}(\theta, t) &= \frac{1}{(4\pi t)^{n/2} \sqrt{\det g(\theta)}} \exp\left(-\frac{d_g^2(\theta, \theta_0)}{4t}\right), n \\ &= \dim(M) = 3 \dots \dots (113) \end{aligned}$$

The Laplace-Beltrami operator on (M, g) is

$$\Delta_g = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j) \dots \dots (114)$$

Spectral decomposition of Δ_g yields eigenfunctions and eigenvalues:

$$\Delta_g \varphi_k = -\lambda_k \varphi_k, 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \dots (115)$$

with orthonormality

$$\int_M \varphi_k(\theta) \varphi_\ell(\theta) \sqrt{\det g(\theta)} d\theta = \delta_{k\ell} \dots \dots (116)$$

The heat kernel admits the expansion

$$K(\theta, \theta', t) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \varphi_k(\theta) \varphi_k(\theta') \dots \dots (117)$$

Minakshisundaram-Pleijel asymptotic expansion (diagonal heat kernel) (Brüning & Heintze, 1984; Bytsenko & Williams, 1999; Colin de Verdière *et al.*,

2021) as $t \downarrow 0$ is

$$K(\theta, \theta, t) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{m=0}^{\infty} a_m(\theta) t^m \dots \dots (118)$$

with

$$a_0(\theta) = 1 \dots \dots (119)$$

$$a_1(\theta) = \frac{1}{6} R(\theta) \dots \dots (120)$$

where $R(\theta) = g^{ij} R_{ij}$ is the scalar curvature and R_{ij} is the Ricci tensor.

The Ricci curvature components are given by

$$\begin{aligned} R_{ij} &= \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^\ell_{ij} \Gamma^k_{\ell k} \\ &\quad - \Gamma^\ell_{ik} \Gamma^k_{j\ell} \dots \dots (121) \end{aligned}$$

The sectional curvature of the plane spanned by tangent vectors u, v is

$$K(u, v) = \frac{R_{ijkl} u^i v^j u^k v^\ell}{(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) u^i v^j u^k v^\ell} \dots \dots (122)$$

where R_{ijkl} is the Riemann curvature tensor.

For $n = 2$ the Gaussian curvature can be written as

$$K_G = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} \dots \dots (123)$$

The Riemannian volume form is

$$dV = \sqrt{\det g(\theta)} d\mu d\sigma dH \dots \dots (124)$$

The total volume of the manifold (when finite) (Figure 4) is

$$V(M) = \int_M \sqrt{\det g(\theta)} d\theta \dots \dots (125)$$

Under a Gaussian approximation the differential entropy of the model at θ satisfies

$$\begin{aligned} S(\theta) &= - \int f(P; \theta) \log f(P; \theta) dP \\ &= \frac{1}{2} \log \det(2\pi e g^{-1}(\theta)) \dots \dots (126) \end{aligned}$$

The Cramér-Rao lower bound for any unbiased estimator $\hat{\theta}$ is the matrix inequality

$$\text{Cov}(\hat{\theta}) \geq g^{-1}(\theta) \dots \dots (127)$$

The natural gradient (Amari's natural gradient) of a scalar objective $J(\theta)$ is

$$\tilde{\nabla} J = g^{-1} \nabla J \dots \dots (128)$$

The Amari-Chentsov third-order tensor (Bravetti *et al.*, 2026; Cheng & Tong, 2026a; Wada & Scarfone, 2026) is

$$T_{ijk} = \mathbb{E}[\partial_i \ell \partial_j \ell \partial_k \ell] \dots \dots (129)$$

From T_{ijk} one defines the family of α -connections

$$\Gamma^{(\alpha)}_{ijk} = \Gamma_{ijk} - \frac{2\alpha}{3} T_{ijk} \dots \dots (130)$$

where $\Gamma_{ijk} = g_{k\ell} \Gamma^\ell_{ij}$ are the metric connection components written with lowered index; for $\alpha = 0$ this reduces to the Levi-Civita connection (Can & Tekin, 2026; Farouk *et al.*, 2023; Manev, 2023; Washburn *et al.*, 2026):

$$\nabla^{(0)} = \text{Levi-Civita} \dots (131)$$

The manifold $(M, g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$ is dually flat when $\alpha = \pm 1$ on appropriate submanifolds; e-flat submanifolds correspond to exponential families while m-flat submanifolds correspond to mixture families.

A divergence function associated to α can be written (one common form) as

$$D^{(\alpha)}(p||q) = \frac{1-\alpha}{4} \left(1 - \int p^{\frac{1-\alpha}{2}} q^{\frac{1+\alpha}{2}} dP \right) \dots (132)$$

which recovers standard divergences for particular α -values (this formula is presented here in a conventional normalized form; other equivalent normalizations appear in the literature).

A Bregman divergence generated by a convex function F is

$$D_F(p||q) = F(p) - F(q) - \langle \nabla F(q), p - q \rangle \dots (133)$$

Finally, a market efficiency index for market i may be defined from pairwise divergences to other markets:

$$\eta_i = 1 + \sum_{j \neq i} w_{ij} D_j(i, j) \dots (134)$$

with weights chosen as heat-kernel (Gaussian) weights on the manifold, for example

$$w_{ij} = \exp\left(-\frac{d_g^2(\theta_i, \theta_j)}{2\sigma^2}\right) \dots (135)$$

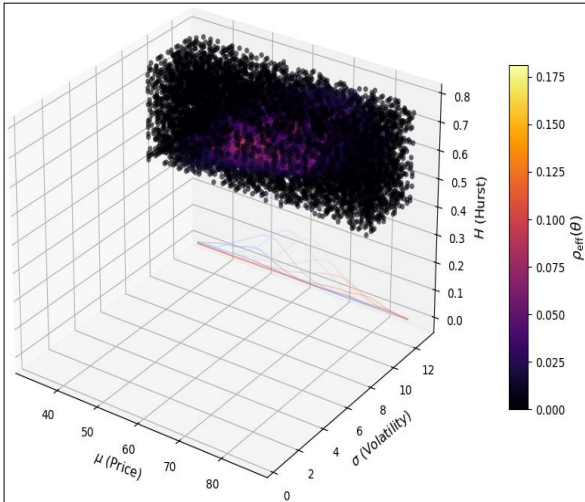


Figure 4: Fisher-Rao metric density in statistical manifold

Koopman operator spectral decomposition of nonlinear market dynamics

The nonlinear price dynamics are given by

$$P_{t+1} = F(P_t, S) \dots (136)$$

with $F: X \rightarrow X$ and state space $X \subset \mathbb{R}^N$ (here $N =$

400 markets). These dynamics are embedded into the infinite-dimensional Hilbert space $H = L^2(X, \mu)$ via the Koopman operator $K: H \rightarrow H$ defined by

$$K\varphi = \varphi \circ F \dots (137)$$

for any observable $\varphi \in H$, where μ is the invariant measure satisfying

$$\mu(F^{-1}(E)) = \mu(E) \text{ for all measurable } E \subset X \dots (138)$$

The Koopman operator is linear:

$$K(c_1\varphi_1 + c_2\varphi_2) = c_1K\varphi_1 + c_2K\varphi_2 \dots (139)$$

and, for measure-preserving F , bounded with

$$\|K\| \leq 1 \dots (140)$$

The eigenvalue problem for K is

$$K\psi_k = \lambda_k\psi_k \dots (141)$$

with $|\lambda_k| \leq 1$, $\psi_k \in H$, and orthonormality

$$\langle \psi_k, \psi_\ell \rangle_\mu = \int_X \psi_k \psi_\ell d\mu = \delta_{k\ell} \dots (142)$$

The eigenvalues are ordered as

$$1 = \lambda_0 \geq |\lambda_1| \geq |\lambda_2| \geq \dots (143)$$

with $\lambda_0 = 1$ corresponding to the invariant density $\psi_0 = 1$.

For any observable φ , the Koopman mode decomposition is

$$\varphi(P_t) = (K^t\varphi)(P_0) = \sum_{k=0}^{\infty} \lambda_k^t b_k \psi_k(P_0) \dots (144)$$

where the Koopman modes are

$$b_k = \langle \varphi, \psi_k \rangle_\mu \dots (145)$$

For vector observables $g: X \rightarrow \mathbb{R}^m$ the decomposition becomes

$$g(P_t) = \sum_{k=0}^{\infty} \lambda_k^t \xi_k \psi_k(P_0) \dots (146)$$

where $\xi_k \in \mathbb{R}^m$ are vector-valued Koopman modes.

Finite-rank approximations are commonly computed using Dynamic Mode Decomposition (DMD). Given snapshot matrices

$$X = [P_0, P_1, \dots, P_{m-1}] \in \mathbb{R}^{N \times m}, Y = [P_1, P_2, \dots, P_m] \in \mathbb{R}^{N \times m} \dots (147)$$

the DMD matrix is

$$K_{\text{DMD}} = YX^\dagger \dots (148)$$

where $X^\dagger$ is the Moore-Penrose pseudoinverse (Bin Jebreen, 2019; Stanojević *et al.*, 2022; Liu *et al.*, 2024; Luo, 2026). The eigen decomposition

$$K_{\text{DMD}} w_k = \lambda_k w_k \dots (149)$$

yields eigenvalues λ_k and eigenvectors w_k . The exact DMD modes are

$$\phi_k = \lambda_k^{-1} Y v_k \dots (150)$$

where v_k are right singular vectors of X . A reconstruction is

$$P(t) \approx \sum_{k=1}^r \lambda_k^t b_k \phi_k \dots \dots (151)$$

with coefficients

$$b_k = \langle P_0, w_k \rangle \dots \dots (152)$$

Extended DMD (EDMD) uses a dictionary of observables $\Phi(P) = [\varphi_1(P), \dots, \varphi_M(P)]^T$. Define the feature matrix

$$\Psi_{ij} = \varphi_j(P_i) \dots \dots (153)$$

Then the EDMD operator approximation is

$$K_{\text{EDMD}} = \Psi^+ \Psi^T \dots \dots (154)$$

where $(\Psi^+)_{ij} = \varphi_j(P_{i+1})$.

Kernel DMD forms kernel (Gram) matrices and uses them to compute the operator. For kernel $k(x, y) = \exp(-\|x - y\|^2 / (2\sigma^2))$ define

$$(K_{xx})_{ij} = k(P_i, P_j), (K_{xy})_{ij} = k(P_i, P_{j+1}) \dots \dots (155)$$

Then

$$K_{\text{kernel}} = K_{xy} K_{xx}^{-1} \dots \dots (156)$$

For continuous-time systems one defines the Koopman generator L by

$$L = \lim_{t \rightarrow 0} \frac{K^t - I}{t} \dots \dots (157)$$

which satisfies

$$L\psi_k = \log(\lambda_k) \psi_k \dots \dots (158)$$

For an SDE

$$dP = F(P) dt + \Sigma(P) dW$$

the infinitesimal generator takes the form

$$L = \sum_{i=1}^N F_i(P) \frac{\partial}{\partial P_i} + \frac{1}{2} \sum_{i,j=1}^N \Sigma_{ij}(P) \frac{\partial^2}{\partial P_i \partial P_j} \dots \dots (159)$$

The spectral expansion of the transfer (Perron-Frobenius) operator P_t is

$$P_t \rho(x) = \sum_{k=0}^{\infty} e^{\Lambda_k t} \langle \rho, \psi_k \rangle_{\mu} \varphi_k(x) \dots \dots (160)$$

where P_t is dual to K^t , $\Lambda_k = \log \lambda_k$ are eigenvalues of L , and φ_k, ψ_k are right/left eigenfunctions satisfying the biorthogonality condition

$$\langle \varphi_k, \psi_{\ell} \rangle_{\mu} = \delta_{k\ell} \dots \dots (161)$$

The Koopman-EDMD algorithm can be summarized as:

- i. collect snapshots $\{P_j\}_{j=0}^m$,
- ii. construct Ψ and Ψ^+ ,
- iii. compute SVD $\Psi = U \Sigma V^*$,
- iv. truncate at rank r ,
- v. compute

$$\tilde{K} = U_r^* \Psi^+ V_r \Sigma_r^{-1} \dots \dots (162)$$

vi. eigen-decompose $\tilde{K} w_k = \lambda_k w_k$,

vii. compute modes $\phi_k = U_r w_k$.

The prediction for a feature vector $z(t)$ is

$$z(t) = \sum_{k=1}^r \lambda_k^t b_k \phi_k \dots \dots (163)$$

with $b = \Phi^+ z_0$.

Hankel-DMD constructs a Hankel matrix from delay embeddings. Form the Hankel matrix

$$H = [h_0, h_1, \dots, h_{m-\tau}] \dots \dots (164)$$

with delay-embedded columns

$$h_j = [g(P_j), g(P_{j+1}), \dots, g(P_{j+\tau-1})]^T,$$

and apply DMD to H .

Spectral properties: the essential spectral radius is defined by

$$r_{\text{ess}}(K) = \lim_{n \rightarrow \infty} \|K^n\|_{\text{ess}}^{1/n} \dots \dots (165)$$

For mixing systems one has $|\lambda_k| < 1$ for $k \geq 1$, and correlations decay according to

$$\begin{aligned} C_{\varphi, \psi}(t) &= \langle \varphi, K^t \psi \rangle_{\mu} - \langle \varphi, 1 \rangle_{\mu} \langle \psi, 1 \rangle_{\mu} \\ &= \sum_{k=1}^{\infty} \lambda_k^t \langle \varphi, \psi_k \rangle_{\mu} \langle \psi_k, \psi \rangle_{\mu} \dots \dots (166) \end{aligned}$$

The mixing time is

$$\tau_{\text{mix}} = -\frac{1}{\log |\lambda_1|} \dots \dots (167)$$

Koopman eigenfunctions provide invariant coordinates: if

$$\psi_k(P_t) = \lambda_k^t \psi_k(P_0) \dots \dots (168)$$

then P_t evolves linearly in the eigenfunction basis. The isosurfaces $\psi_k(P) = c$ partition state space into invariant (or approximately invariant) sets.

For 3D visualization (Figure 5) we project onto coordinates (P, V, I) and compute the dominant eigenfunction ψ_1 via EDMD using polynomial observables

$$\varphi_{ijk}(P, V, I) = P^i V^j I^k \text{ for } i + j + k \leq 3 \dots \dots (169)$$

The feature matrix is

$$\Psi_{n,\ell} = \varphi_{\ell}(x_n), x_n = (P_n, V_n, I_n) \dots \dots (170)$$

(When using EDMD the matrix approximation is computed as above.) The eigenfunction is reconstructed as

$$\psi_1(x) = \sum_{\ell=1}^M v_{1,\ell} \varphi_{\ell}(x) \dots \dots (171)$$

where v_1 is the eigenvector of the EDMD matrix K .

Iso-surface extraction uses marching cubes on a regular grid $G \subset \mathbb{R}^3$ with values $\psi_1(x_{ijk})$ at grid points. The level set

$$S_c = \{x \in \mathbb{R}^3: \psi_1(x) = c\} \dots \dots (172)$$

is approximated by a triangulated mesh M_c with vertices v_p

and faces F_q .

The mesh Laplacian L_{pq} is defined by

$$L_{pq} = \begin{cases} \deg(p) & p = q, \\ -1 & (p, q) \in E \dots \dots (173) \\ 0 & \text{otherwise,} \end{cases}$$

where E denotes the set of mesh edges.

The discrete (angle-based) curvature at vertex p is

$$K_p = 2\pi - \sum_{q \sim p} \theta_{pq} \dots \dots (174)$$

where θ_{pq} are the angles around vertex p between adjacent faces. The Gaussian curvature integrated over the surface is

$$K_G = \frac{1}{A} \int_{M_c} K dA \dots \dots (175)$$

with A the total area of M_c . The mean curvature is

$$H = \frac{1}{2}(k_1 + k_2) \dots \dots (176)$$

where k_1, k_2 are the principal curvatures. The Willmore energy is

$$W = \int_{M_c} H^2 dA \dots \dots (177)$$

Spectral mesh decomposition uses eigenfunctions of the mesh Laplacian:

$$L f_k = \mu_k f_k \dots \dots (178)$$

with

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \dots (179)$$

The mesh Fourier coefficients (Rysbauly *et al.*, 2022) are

$$\hat{f}^k = \langle f, f_k \rangle_A \dots \dots (180)$$

where A denotes the diagonal area matrix used for the inner product. The reconstruction is

$$f = \sum_{k=0}^{\infty} \hat{f}^k f_k \dots \dots (181)$$

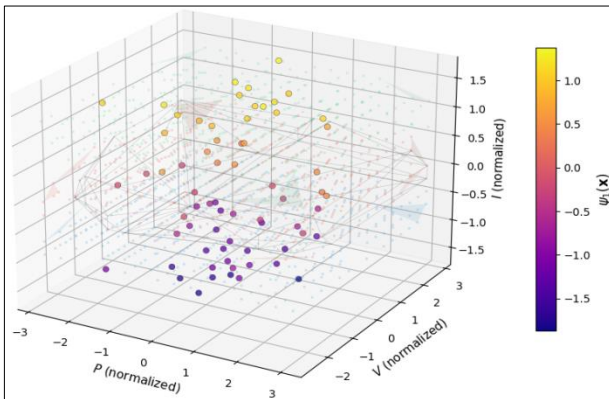


Figure 5: Koopman eigenfunction isosurfaces of market modes

Percolation theory and network contagion in rural market hierarchies

We model the rural market system as a random geometric graph $G = (V, E)$ where the vertex set $V = \{1, \dots, N\}$ represents $N = 400$ markets and the edge set is

$$E = \{(i, j) : |P_i - P_j| < \delta \dots \dots (182)$$

with Euclidean geographic distance

$$d_{\text{geo}}(i, j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \dots \dots (183)$$

The occupation (connection) probability as a function of price threshold δ is

$$p(\delta) = \frac{1}{N(N-1)} \sum_{i \neq j} \mathbf{1}(|P_i - P_j| < \delta) \mathbf{1}(d_{\text{geo}}(i, j) < R_c) \dots \dots (184)$$

The order parameter is the relative size of the largest connected component in the thermodynamic limit,

$$P_{\infty}(p) = \lim_{N \rightarrow \infty} \frac{|C_{\text{max}}|}{N} \dots \dots (185)$$

where C_{max} is the largest connected component. Near the critical threshold p_c ,

$$P_{\infty}(p) \sim (p - p_c)^{\beta} \text{ as } p \rightarrow p_c^+ \dots \dots (186)$$

with the two-dimensional percolation exponent $\beta = \frac{5}{36}$

The correlation length diverges as

$$\xi(p) \sim |p - p_c|^{-\nu} \dots \dots (187)$$

with $\nu = \frac{4}{3}$. The susceptibility, defined via the cluster-size distribution $n_s(p)$ (number of clusters of size s), behaves as

$$\chi(p) = \frac{1}{N} \sum_s s^2 n_s(p) \sim |p - p_c|^{-\gamma} \dots \dots (188)$$

with $\gamma = \frac{43}{18}$.

The fractal dimension of the percolating cluster at criticality is

$$D_f = 2 - \frac{\beta}{\nu} = \frac{91}{48} \dots \dots (189)$$

Cluster-number scaling follows

$$n_s(p) \sim s^{-\tau} f((p - p_c)s^{\sigma}) \dots \dots (190)$$

with $\tau = \frac{187}{91}$ and $\sigma = \frac{36}{91}$

Hyper-scaling relations in dimension d (here $d = 2$) include

$$D_f = d - \frac{\beta}{\nu} \dots \dots (191)$$

$$\gamma + 2\beta = d\nu \dots \dots (192)$$

$$2 - \alpha = d\nu \dots \dots (193)$$

Markets are stratified into $L = 4$ socioeconomic layers by income quartiles. Define

$$V_{\ell} = \{i : Q_{\ell-1} \leq I_i < Q_{\ell}\}, \ell = 1, \dots, 4 \dots \dots (194)$$

where Q_{ℓ} are the quartile boundaries and I_i is market i 's

income. For each layer ℓ the intra-layer occupation probability is

$$p_\ell(\delta) = \frac{1}{N_\ell(N_\ell - 1)} \sum_{i \neq j \in V_\ell} \mathbf{1}(|P_i - P_j| < \delta) \dots \dots (195)$$

The layer-specific percolation threshold $p_c^{(\ell)}$ satisfies

$$P_\infty^{(\ell)}(p_c^{(\ell)}) = 0 \dots \dots (196)$$

A renormalization-group (RG) approach defines block variables on blocks B_l of linear size b . The block-averaged price (block "spin") is

$$P_l^{(b)} = b^{-d} \sum_{i \in B_l} P_i \text{ (with } d = 2) \dots \dots (197)$$

The RG transformation R_b maps $p \mapsto p'$. A fixed point p^* satisfies $p^* = R_b(p^*)$; linearizing near p^* ,

$$p' - p^* \approx \Lambda_b(p - p^*), \Lambda_b = b^{1/\nu} \dots \dots (198)$$

Thus

$$\nu = \frac{\log b}{\log \Lambda_b} \dots \dots (199)$$

The Harris criterion states disorder is relevant if $d\nu < 2$. For 2D percolation (Figure 6)

$$d\nu = \frac{8}{3} > 2 \dots \dots (200)$$

so weak disorder is irrelevant. Network contagion is modeled with SIR dynamics on G , stratified by node degree k . Let $S_k(t), I_k(t), R_k(t)$ be the fractions of susceptible, infected, and recovered nodes of degree k . The mean-field rate equations are

$$\frac{d}{dt} S_k = -\lambda k S_k \Theta \dots \dots (201)$$

$$\frac{d}{dt} I_k = \lambda k S_k \Theta - I_k \dots \dots (202)$$

$$\frac{d}{dt} R_k = I_k \dots \dots (203)$$

where λ is the infection rate (recovery rate set to 1 by time scaling), and

$$\Theta = \sum_k \frac{kP(k)}{\langle k \rangle} I_k \dots \dots (204)$$

is the probability that a randomly chosen neighbor is infected.

The basic reproduction number is

$$R_0 = \lambda \frac{\langle k^2 \rangle}{\langle k \rangle} \dots \dots (205)$$

so the epidemic threshold is

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle} \dots \dots (206)$$

For scale-free networks with $P(k) \sim k^{-\gamma}$ and

$$2 < \gamma \leq 3 \dots \dots (207)$$

$\langle k^2 \rangle \rightarrow \infty$, hence $\lambda_c \rightarrow 0$ and no finite epidemic threshold exists.

Mapping SIR to bond percolation gives bond occupation probability

$$p = 1 - e^{-\lambda} \dots \dots (208)$$

The giant component size equals the final epidemic size

$$R_\infty = 1 - \sum_k P(k) e^{-\lambda k \phi} \dots \dots (209)$$

where ϕ satisfies the self-consistent equation

$$\phi = 1 - \sum_k \frac{kP(k)}{\langle k \rangle} e^{-\lambda k \phi} \dots \dots (210)$$

Using generating functions,

$$G_0(z) = \sum_k P(k) z^k \dots \dots (211)$$

$$G_1(z) = \frac{G'_0(z)}{G'_0(1)} \dots \dots (212)$$

The mean component size (finite clusters) can be expressed via derivatives of the generating functions; evaluated at $z = 1$ one finds

$$\langle s \rangle = 1 + \frac{G'_0(1)}{1 - G'_1(1)} \dots \dots (213)$$

The percolation condition for the emergence of a giant component is

$$G'_1(1) = 1 \dots \dots (214)$$

For the random geometric graph with connection function $g(r) = \mathbf{1}(r < R_c)$, the degree distribution is approximately Poisson with mean

$$\langle k \rangle = \rho \pi R_c^2 \dots \dots (215)$$

where $\rho = N/A$ is the node density (nodes per unit area). The continuum percolation threshold is commonly given in terms of the dimensionless parameter $\eta = \rho \pi R_c^2$; for 2D

$$\eta_c \approx 4.512 \dots \dots (216)$$

The Molloy-Reed criterion for giant-component existence is

$$\langle k^2 \rangle - 2\langle k \rangle > 0 \dots \dots (217)$$

The configuration model generates networks with a prescribed degree sequence. The Watts-Strogatz model (Song & Wang, 2014; Maier, 2019; Merenda & Bruno, 2023; Wang *et al.*, 2026b) interpolates between a regular lattice ($p_{\text{rew}} = 0$) and a random graph

$$p_{\text{rew}} = 1 \dots \dots (218)$$

showing a small-world regime at intermediate p_{rew} . The clustering coefficient C is defined as

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triples}} \dots \dots (219)$$

For random geometric graphs in 2D one finds an approximate value

$$C \approx 1 - \frac{4\sqrt{3}}{3} \approx 0.5865 \dots \dots (220)$$

Betweenness centrality of a vertex v is

$$BC(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}} \dots \dots (221)$$

where σ_{st} is the number of shortest paths from s to t and $\sigma_{st}(v)$ is the number of those paths passing through v .

The percolation backbone is the set of edges that carry current in the infinite cluster. Its fractal dimension in 2D is

$$D_B \approx 1.643 \dots \dots (222)$$

The red bonds (singly connected edges) have fractal dimension

$$D_{\text{red}} = \frac{1}{\nu} = \frac{3}{4} \dots \dots (223)$$

The chemical distance (shortest-path length along the cluster) scales as

$$\ell \sim r^{D_{\min}}, D_{\min} \approx 1.130 \dots \dots (224)$$

The (transport) conductivity exponent $\tilde{\mu}$ obeys the relation $\tilde{\mu} = (d-2)\nu + \beta$; for $d = 2$ this gives

$$\tilde{\mu} = \frac{4}{3} \dots \dots (225)$$

Universality: the universality class is fixed by dimension and symmetry; all short-range 2D percolation problems share the same critical exponents. The conformal-field-theory (CFT) description of 2D percolation yields central charge $c = 0$ and scaling dimensions

$$\Delta_\sigma = \frac{5}{48}, \Delta_\varepsilon = \frac{5}{4} \dots \dots (226)$$

Cluster boundaries in critical percolation are described by Schramm-Loewner evolution (Lawler, 2007) with parameter

$$\kappa = 6 \dots \dots (227)$$

The crossing probability for a rectangle of aspect ratio $L \times W$ can be written in terms of elliptic functions; with $q = e^{-\pi K'/K}$ (where K, K' are complete elliptic integrals with modulus k satisfying $K'/K = L/W$) the crossing probability has a product representation

$$P_{\text{cross}} = \prod_{n=1}^{\infty} \frac{1 + q^{2n-1}/2}{1 - q^{2n-1}/2} \dots \dots (228)$$

Cardy's formula for the critical crossing probability as a function of the aspect ratio $r = L/W$ can be written using hypergeometric functions. With $\eta = \left(\frac{1+e^{-\pi r}}{1-e^{-\pi r}}\right)^2$,

$$P_{\text{cross}}(r) = \frac{\Gamma(1/3)}{2^{2/3}\Gamma(2/3)} \eta^{1/3} {}_2F_1(1/3, 2/3; 4/3; \eta) \dots \dots (229)$$

To visualize percolation across socioeconomic strata, define for each layer ℓ the surface

$$S_\ell(\delta, t) = \frac{1}{N_\ell} \sum_{i \in V_\ell} \mathbf{1}(i \in C_{\max}(\delta, t)) \dots \dots (230)$$

the fraction of layer ℓ nodes belonging to the largest component at price threshold δ and time t . Layer separation

$$\Delta S = S_{\ell+1} - S_\ell \dots \dots (231)$$

measures inequality in market connectivity. The Gini coefficient for percolation-layer shares can be computed as

$$G = \frac{2}{\bar{S}L^2} \sum_{\ell, m} |S_\ell - S_m|, \bar{S} = \frac{1}{L} \sum_{\ell} S_\ell \dots \dots (232)$$

The Theil index is

$$T = \frac{1}{L} \sum_{\ell} \frac{\bar{S}}{S_\ell} \log \frac{\bar{S}}{S_\ell} \dots \dots (233)$$

The Lorenz curve is

$$L(x) = \frac{1}{\bar{S}} \int_0^x S(y) dy, x = \frac{\ell}{L} \dots \dots (234)$$

Network robustness is measured by the R-value

$$R = \frac{1}{N} \sum_{q=1}^N s(q) \dots \dots (235)$$

where $s(q)$ is the fraction of nodes in the largest component after removing a fraction q of nodes (typically removed in descending degree order).

The percolation transition in 2D is continuous (second-order). The specific-heat exponent is

$$\alpha = 2 - d\nu = -\frac{2}{3} \dots \dots (236)$$

The Fisher exponent is

$$\eta = \frac{5}{24} \dots \dots (237)$$

The gap exponent is

$$\Delta = \beta + \gamma \approx 2.389 \dots \dots (238)$$

A hyperscaling-like relation is written as $d\rho = 2 - \alpha$ with $\rho = \beta\delta/(\delta + 1)$. For 2D percolation

$$\delta \approx \frac{91}{5} = 18.2 \dots \dots (239)$$

(quoted as 18.17 in some references). The spanning probability at criticality for finite system size L obeys finite-size scaling,

$$P_{\text{span}}(L) = f((p - p_c)L^{1/\nu}) \dots \dots (240)$$

and the finite-size scaling ansatz for the order parameter is

$$P_\infty(p, L) = L^{-\beta/\nu} \tilde{P}((p - p_c)L^{1/\nu}) \dots \dots (241)$$

The Binder cumulant for the largest-cluster magnitude is

$$U_L = 1 - \frac{3\langle |C_{\max}|^2 \rangle}{2\langle |C_{\max}|^4 \rangle} \dots \dots (242)$$

which has the property of crossing at p_c for different system sizes L .

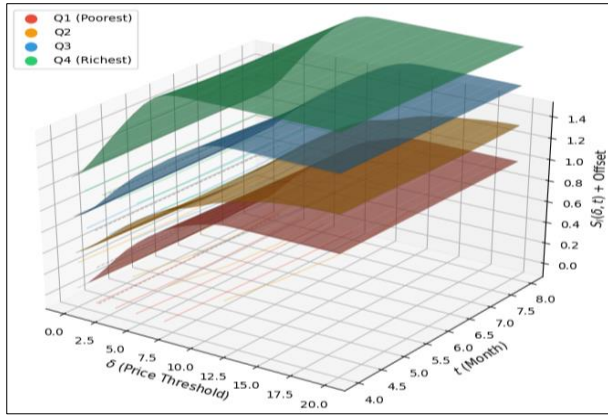


Figure 6: Percolation threshold surfaces across socioeconomic strata

Conclusion and perspectives

This paper presents a six-layer mathematical synthesis for rural off-season price dynamics.

Layer I. Fractional stochastic reaction-diffusion-advection

Price density $P = P(x, t)$ evolves under a stochastic fractional reaction-diffusion-advection equation with multiplicative noise and an anisotropic diffusion tensor:

$$\partial_t P = -D(-\nabla^2)^{\alpha/2} P - v \cdot \nabla P + R(P; S) + \sigma(P; S) \partial_t W \dots \dots (243)$$

where the anisotropic diffusion tensor is

$$D = \text{diag}(D_0 e^{-\beta_1 T_{\text{trans}}}, D_0 e^{-\beta_2 M}) \dots \dots (244)$$

the fractional Laplacian $(-\nabla^2)^{\alpha/2}$ has symbol $|k|^\alpha$ with anomalous transport exponent $\alpha \in (1, 2)$, and the multiplicative noise is modeled as

$$\sigma(P; S) = \sigma_0 P^\nu (1 + \eta \frac{W_{\text{weather}}}{S_{\text{income}}}) \dots \dots (245)$$

Layer II. Coupled fractional Brownian motion and fractional Liouville equation

A network of regional prices $\{P_i\}$ follows coupled fractional Brownian motion (fBm) dynamics:

$$dP_i = \mu_i dt + \sigma_i dB^{H_i}(t) + \sum_{j \in N(i)} J_{ij}(P_j - P_i) dt \dots \dots (246)$$

where the Mandelbrot-van Ness representation of fBm is

$$B^{H_i}(t) = \frac{1}{\Gamma(H_i + \frac{1}{2})} \int_{-\infty}^t [(t-s)^{H_i-\frac{1}{2}} - (-s)^{H_i-\frac{1}{2}}] dW(s) \dots \dots (247)$$

and long-range covariance satisfies

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \dots \dots (248)$$

On the level of probability densities this yields a

fractional Liouville (fractional Fokker-Planck) equation with Caputo time derivative:

$$\partial_t^\beta \rho = -\nabla_\Psi \cdot (F\rho) + \frac{1}{2} \nabla_\Psi^2 : (\Sigma \Sigma^T \rho) \dots \dots (249)$$

where the Caputo derivative is

$$\partial_t^\beta \rho = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \partial_s \rho(s) ds \dots \dots (250)$$

Layer III. Catastrophe theory and bifurcations

A cusp-type potential models abrupt transitions in price:

$$V(P; a, b) = \frac{1}{4} P^4 + \frac{1}{2} a P^2 + b P \dots \dots (251)$$

with control parameters

$$a = a_0 - a_1 \frac{S_{\text{income}}}{S_{\text{storage}}} + a_2 W_{\text{weather}}, b = b_0 - b_1 M + b_2 \nabla^2 P \dots \dots (252)$$

The bifurcation (cusp) set is

$$4a^3 + 27b^2 = 0 \dots \dots (253)$$

Near bifurcation the amplitude equation for the slow mode u takes the form

$$\partial_t u = \varepsilon_1 u + \varepsilon_2 u^2 - u^3 + D_c \nabla^2 u \dots \dots (254)$$

and stream-tube parameterization of bifurcation flows uses

$$T(s, \theta, r) = \gamma(s) + r \cos(\theta) n_1(s) + r \sin(\theta) n_2(s) \dots \dots (255)$$

with a Bishop frame satisfying

$$\frac{d}{ds} e_1 = -k_1 t, \frac{d}{ds} e_2 = -k_2 t \dots \dots (256)$$

Layer IV. Information geometry on a statistical manifold

Model prices by a parametric density $f(P; \theta)$ (e.g., generalized hyperbolic family). The Fisher-Rao metric is

$$g_{ij}(\theta) = -\mathbb{E}_\theta[\partial_i \partial_j \ell(\theta)] \dots \dots (257)$$

where $\ell(\theta) = \log f(P; \theta)$. Geodesics satisfy

$$\frac{d^2 \theta^k}{dt^2} + \Gamma_{ij}^k \frac{d\theta^i}{dt} \frac{d\theta^j}{dt} = 0 \dots \dots (258)$$

with Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \dots \dots (259)$$

The Kullback-Leibler divergence between two densities is

$$D_{\text{KL}}(f_i \| f_j) = \int f_i \log \frac{f_i}{f_j} dP \dots \dots (260)$$

and the short-time heat kernel (effective density) on the manifold is approximated by

$$\rho_{\text{eff}}(\theta, t) = (4\pi t)^{-\frac{n}{2}} (\det g)^{-1/2} \exp(-\frac{d_g^2}{4t}) \dots \dots (261)$$

where d_g is the geodesic distance and $R = g^{ij} R_{ij}$ denotes scalar curvature.

Layer V. Koopman operator spectral theory and modal decompositions

The Koopman operator K acts on observables ϕ by composition $K\phi = \phi \circ F$. Eigenfunctions and eigenvalues solve

$$K\psi_k = \lambda_k \psi_k, |\lambda_k| \leq 1 \dots (262)$$

A finite-dimensional reconstruction using r modes is

$$P(t) \approx \sum_{k=1}^r \lambda_k^t b_k v_k \dots (263)$$

In Extended Dynamic Mode Decomposition (EDMD) the matrix approximation is

$$K \approx \Psi^+ \Psi \dots (264)$$

while the generator of continuous dynamics takes the form

$$\mathcal{L} = \sum_i F_i \partial_{P_i} + \frac{1}{2} \sum_{i,j} \Sigma_{ij} \partial_{P_i} \partial_{P_j} \dots (265)$$

Iso-surface extraction uses

$$\mathcal{S}_c = \{x: \psi_1(x) = c\} \dots (266)$$

with mesh Laplacian entries L_{pq} and Willmore energy

$$W = \int H^2 dA \dots (267)$$

where H is mean curvature.

Layer VI. Percolation theory on random geometric graphs

Construct a price similarity graph $G = (V, E)$ with occupation probability based on pairwise price closeness and geographic proximity:

$$p(\delta) = \frac{1}{N(N-1)} \sum_{i \neq j} \mathbf{1}(|P_i - P_j| < \delta) \mathbf{1}(d_{\text{geo}} < R_c) \dots (268)$$

The giant-component order parameter near criticality scales as

$$P_\infty(p) \sim (p - p_c)^\beta \dots (269)$$

with percolation critical exponent $\beta = \frac{5}{36}$, correlation length

$$\xi(p) \sim |p - p_c|^{-\nu}, \nu = \frac{4}{3} \dots (270)$$

susceptibility

$$\chi(p) \sim |p - p_c|^{-\gamma}, \gamma = \frac{43}{18} \dots (271)$$

and fractal dimension

$$D_f = 2 - \frac{\beta}{\nu} = \frac{91}{48} \dots (272)$$

SIR-type contagion on the network for nodal infection fraction ρ_k follows

$$\frac{d\rho_k}{dt} = -\rho_k + \lambda_{\text{eff},k}(1 - \rho_k) \Theta(\lambda_{\text{eff}}) \dots (273)$$

with epidemic threshold

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle} \dots (274)$$

Stacked percolation surfaces across income quartiles V_ℓ are defined by

$$S_\ell(\delta, t) = \frac{1}{N_\ell} \sum_{i \in V_\ell} \mathbf{1}(i \in \mathcal{C}_{\max}(\delta, t)) \dots (275)$$

Integrated interpretation

The integration shows rural price volatility arises from interacting mechanisms: fractional transport with anomalous exponent $\alpha \in (1, 2)$ (Layer I), catastrophic transitions across the bifurcation manifold

$$\mathcal{B} = \{(a, b): 4a^3 + 27b^2 = 0\} \dots (276)$$

geometric information divergence on a Riemannian manifold $(\mathcal{M}, g)$ with scalar curvature $R = g^{ij} R_{ij}$ (Layer IV), operator-theoretic modal decomposition via Koopman eigenfunctions ψ_k and eigenvalues λ_k encoding decay rates (Layer V), and network percolation with universal critical exponents $(\beta, \nu, \gamma) = (\frac{5}{36}, \frac{4}{3}, \frac{43}{18})$ (Layer VI).

Visualizations

Six complementary visualizations elucidate multi-scale structure: 3D choropleth of spatial price elevation; 4D glyph plot of socioeconomic state vectors (P, V, I, H) ; stream-tube plots of bifurcation flows in (a, b, P) space; Fisher–Rao metric density $\rho_{\text{eff}}(\theta)$ on the statistical manifold; Koopman eigenfunction isosurfaces $\psi_1(x) = c$ with mesh overlay; and stacked percolation threshold surfaces $S_\ell(\delta, t)$ across socioeconomic strata.

Policy implications

The mathematical structure suggests policy levers and diagnostics:

i. Increasing storage S_{storage} shifts control parameter $a = a_0 - a_1 S_{\text{storage}}/S_{\text{income}} + a_2 W_{\text{weather}}$ away from the cusp set $\{4a^3 + 27b^2 = 0\}$, reducing catastrophic price jumps. (See (252) and (253).)

ii. Income stabilization contracts Fisher–Rao geodesic distances $d_g(\theta_i, \theta_j)$, promoting information efficiency and convergence toward the invariant measure μ .

iii. Regulating market connectivity adjusts the percolation threshold p_c and the epidemic reproduction number $R_0 = \lambda \langle k^2 \rangle / \langle k \rangle$, mitigating systemic contagion.

iv. Mapping the spatial Hurst exponent field $H(x)$ locates regions of persistent volatility for targeted intervention (Layer II).

v. The Koopman spectral gap $1 - |\lambda_1|$ quantifies intrinsic mixing rates and the timescale over which policies will affect dynamics (Layer V).

Future directions

Possible extensions include:

i. Variable-order fractional Laplacian $(-\nabla^2)^{\alpha(x)/2}$ to capture spatially heterogeneous transport infrastructure.

- ii. Incorporating Lévy flights with heavy-tailed jump measure $\nu(dy) \sim |y|^{-d-\alpha} dy$ for extreme-event modeling.
- iii. Developing deep-Koopman architectures using neural-network observables $\varphi_j(P; \theta_{NN})$ for high-dimensional embeddings.
- iv. Applying conformal bootstrap methods (from $c = 0$ minimal models) to compute percolation observables non-perturbatively.

Generalizability

The framework generalizes to other agricultural commodities, climate adaptation scenarios, and developing-economy market structures where fractional dynamics, critical transitions, information asymmetry, operator modes, and network contagion jointly determine socioeconomic outcomes.

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