**η-Duals of Some Double Sequence Spaces**

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Abstract

P. Chandra and B.C. Tripathy [13] have generalized the notion of the köthe-toeplitz dual of sequence spaces on introducing the concept of η-dual of order r, for r ≥ 1 of sequence spaces. B.C. Tripathy and B. Sharma [3] have introduced the notion of η-dual of order r, for 0 ≤ r ≤ 1 of sequence spaces. Ansari and Gupta [1] have generalized the notion of the köthe-Toeplitz dual of sequence spaces on introducing the concept of η-dual of order r, for 0 < r ≤ 1 of sequence spaces. In this paper, we have defined and determined the η-dual of some double sequence spaces for 0 < r ≤ 1 and have established their perfectness in relation to η-dual for 0 < r ≤ 1.

**Keywords:** α-dual, η-dual, perfect space, Double sequence, Bounded variation, Regular convergent, 2/lr-space.

INTRODUCTION

A sequence space is defined to be a linear space of sequences as its element with respect to the coordinate wise addition and scalar multiplication. It is a scalar sequence space or a vector sequence space according as the sequences consists of scalar (real or complex) or vectors taken from a vector space. A sequence of the form \((a_i)\) \(_{i=1}^\infty\) will be called a single sequence and a sequence of the form \((a_{m,n})\) \(_{m,n=1}^\infty\) will be called a double sequence or a matrix.

Köthe and Toeplitz [8] introduced the idea of dual sequence space, whose main results concerned with i-duals. Later on it was studied by P. Chandra and B.C. Tripathy [13] and Gupta and Gupta [7], Maddox [10], Lascarides [9], Okutoyi [12] and many others. P. Chandra and B.C. Tripathy [13] have generalized the notion of α-duals on introducing the notion of η-duals of order r, for r ≥ 1 of sequence spaces and Ansari & Gupta [1], have generalized the notion of the α-duals on introducing the notion of η-duals of order r, for 0 < r ≤ 1 of sequence spaces.

Browmich [04] introduced the notion of double sequence spaces and Hardy [6] introduced the notion of bounded variation double sequence spaces. Later on it was studied by P. Chandra and B.C. Tripathy and B. Sarma [3], Basarir and Sonalean [2], Tripathy, Choudhary and Sharma [14], Moricz [11] and many others.

In this paper, the space of all, bounded, convergent in Pringsheim’s sense, regularly null, absolutely summable, p-absolutely summable finite, bounded variation regularly convergence, Null in Pringsheim’s sense, eventually alternating and strongly p-cesaro summable double sequence spaces are denoted by \(2_{c0}, 2_{c0}^\circ, 2_c, 2_c^R, 2_c^0, 2_c^R, 2_c^0, 2_l, 2_{l0}, 2_{b0}, 2_{b}, 2_{b0}, 2_{b}, 2_{b0}, 2_{b}, 2_p, 2_p^p, 2_p^0\) respectively and a double sequence is denoted by \((a_{mn})\) or \((x_{mn})\) respectively.

List of some double sequence spaces, whose η-dual will be obtained in this paper are:

1. \(2_c^0\) = \(\{ < a_{mn} > \in 2_c : a_{mn} \to 0 \text{ as } \min (m,n) \to \infty \}\)
2. \(2_c^0\) = \(\{ < a_{mn} > \in 2_c : a_{mn} \to 0 \text{ as } \max (m,n) \to \infty \}\)
3. \(2_c \) = \(\{ < a_{mn} > \in 2_c : \lim_{m,n \to \infty} a_{mn} = L_m, \text{ where } L_m \in c \text{ for each } m \in N \}\)
4. \(2_c^\infty \) = \(\{ < a_{mn} > \in 2_c : \sup_{m,n} | a_{mn} | < \infty \}\)
5. \(2_l \) = \(\{ < a_{mn} > \in 2_l : \sum_{m,n} a_{mn} r \to \infty, \text{ where } r \text{ is a real no. such that } 0 < r < \infty \}\)
6. \(2_{b0} \) = \(\{ < a_{mn} > \in 2_{b0} : \sum_{m} a_{mn} = \infty, \Sigma \Delta a_{mn} = \infty \}\)
7. \(2_{b0}^R\) = \(\{ < a_{mn} > \in 2_{b0} : \sum_{m,n} a_{mn} r \to \infty, \text{ where } r \text{ is a real no. such that } 0 < r < \infty \}\)

where \(\Delta a_{mn} = a_{m,n+1} - a_{m,n} \) and \(\Delta a_{m,n} = a_{m,n+1} - a_{m,n} \) and \(\Delta a_{mn} = a_{m,n+1} - a_{m,n} \) and \(\Delta a_{mn} = a_{m,n+1} - a_{m,n} \)

8. \(2_{p0}\) = \(\{ < a_{ij} > \in 2_{p0} : \lim_{m,n \to \infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \infty \}\)

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\[ 2\sigma = \{ (a_{mn}) \in 2\omega : a_{mn} = -a_{m+1,n} \text{ for all } n \geq n^0 \text{ and } a_{mn} = -a_{m,n+1} \text{ for all } m \geq m^0 \} \]

The spaces \( 2c, 2c_0, 2c_1, 2c \cap 2p_0, 2c_0 \cap 2p_0 \) and \( 2p_0 \) are normed linear spaces with the norm given by

\[ ||a_{nk}|| = \sup_{m,n} |a_{mn}| \]

from the above definition, it is clear that

\[ 2c \subset 2c_0 \supseteq 2p_0 \subset 2c_0 \supseteq 2p_0 \]

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\[ \sup_{m,n} |a_{mn}| \]

Theorem 2. Let \( E \) be a non-empty subset of \( 2\omega \) and \( r \geq 1 \). Then \( \eta \)-dual of order \( r \) of \( E \) is denoted by \( E^{(r)} \) and defined by B.C. Tripathy and B. Sarma [3] as

\[ E^{(r)} = \{ (a_{mn}) \in 2\omega : \sum_{m,n} |a_{mn} x_{mn}|^{1/r} < \infty \text{ for all } (x_{mn}) \in E \} \]

Taking \( r = 1 \) in above definition, we also get the \( \alpha \)-dual of \( E \).

A non-empty subset \( E \) of \( \omega \) is said to be perfect or \( \eta \)-reflexive with respect to \( \eta \)-dual if \( E^{\eta \eta} = (E^{(1)})^{\eta} = E \).

Definition 3. Let \( E \) be a non-empty subset of \( 2\omega \) and \( 0 < r \leq 1 \); then we [1] define the \( \eta \)-dual of order \( r \) of \( E \) as

\[ E^{(r)} = \{ (a_{mn}) \in 2\omega : \sum_{m,n} |a_{mn} x_{mn}|^{1/r} < \infty \text{ for all } (x_{mn}) \in E \} \]

Taking \( r = 1 \) in above definition, we also get the \( \alpha \)-dual of \( E \). Also, a non-empty subset \( E \) of \( \omega \) is said to be perfect or \( \eta \)-reflexive with respect to \( \eta \)-dual if \( E^{\eta \eta} = E \). In this paper, the sum without limit means that the summation is from \( m = 1 \) to \( \infty \) and \( n = 1 \) to \( \infty \).

**MAIN RESULTS**

The proof of the following Lemma is obvious in view of the definition of \( \eta \)-dual of double sequences.

**Lemma (1):**

1. \( E^{\eta \eta} \) is a linear subspace of \( 2\omega \) for every \( E \subset 2\omega \).
2. \( E \subset F \) implies \( E^{\eta} \supset E^{\eta} \).
3. \( E \subset E^{\eta \eta} \) for every \( E \subset 2\omega \).

**Theorem 1.** \( (2p_r)^{\eta} = 2p_0 \) and \( (2p_0)^{\eta} = 2p_r \). The spaces \( 2p_r \) and \( 2p_0 \) are perfect spaces, where \( 0 < r \leq 1 \).

**Proof:** First, we shall show that \( (2p_r)^{\eta} = 2p_0 \).

where, \( (2p_r)^{\eta} = (a_{mn}) \in 2\omega : \sum_{m,n} |a_{mn} x_{mn}|^{1/r} < \infty \text{ for all } (x_{mn}) \in (2p_r)^{\eta} \}

Let \( x_{mn} \in (2p_0)^{\eta} \) and \( (x_{mn}) \in (2p_r)^{\eta} \).

\[ \sup_{m,n} |a_{mn}| < \infty \text{ and } \sum_{m,n} |a_{mn} x_{mn}|^{1/r} \]

Therefore, the spaces \( 2p_0 \) and \( 2p_r \) are perfect.

**Theorem 2.** \( (2c_0)^{\eta} = (2c_0)^{R} = 2p_r \). The spaces \( 2c_0 \) and \( 2c_0^{R} \) are not perfect, where \( 0 < r \leq 1 \).

**Proof:** Since \( 2c_0^{R} \subset 2c_0^{R} \subset 2p_0 \subset 2p_0 \),

By Lemma 1 (ii) and theorem (1),
\[ 2^r = (2^{\infty})^\eta \subset (2^r)^{\infty} \subset (2^r)^\eta \subset (2^0)^{\infty} \]

Hence, in order to prove the theorem, it is sufficient to show that \((2^0)^{\infty} \subset 2^r\).

Let \(a_{mn} \notin 2^r \).

Then we can find sequences \(< m^i >\) and \(< n^i >\) of natural numbers with \(m^0 = 1, n^0 = 1\) such that

\[ \sum_{m=1}^{m_i} \sum_{n=1}^{n_i} a_{mn} l^r > \frac{1}{(i+1)^{r/2}}, \quad i = 0, 1, 2, 3, \ldots. \]

Define a sequence \((x_{mn})\) by

\[ x_{mn} = (i+1) \cdot \frac{1}{3} \]

for \(m^i-1 < m \geq m^i\) and \(n^i-1 < n \leq n^i\), for all \(i \in \mathbb{N}\).

Then \((x_{mn}) \in 2^c^{\infty}

Now,

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} x_{mn} l^r = \sum_{i=0}^{\infty} \left( \sum_{m=1}^{m_i} \sum_{n=1}^{n_i} a_{mn} x_{mn} l^r - \sum_{m=1}^{m_{i+1}} \sum_{n=1}^{n_{i+1}} a_{mn} x_{mn} l^r \right) \]

\[ = \sum_{i=0}^{\infty} \frac{1}{(i+1)^{r/2}} \cdot \frac{1}{3} = \frac{1}{3} \sum_{i=0}^{\infty} \frac{1}{(i+1)^{r/2}} = \infty \text{ (because } 0 < \frac{5r}{6} < 1) \]

Thus \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} x_{mn} l^r = \infty\).

Then \((a_{mn}) \notin (2^c)^{\infty}\)

Hence we have \((2^c)^{\infty} \subset 2^r\)

Thus, it is proved that

\[ (2^{c^r})^\eta = (2^{c^r})^{\infty} \subset 2\]

Further more,

since \((z^w)^{\infty} = (z^w)^\eta = (l_{z^w} + z)\)

and \((z^{c^r})^\eta = (z^{c^r})^{\infty} = (l_{z^{c^r}} + z)\)

Therefore, the spaces \(2^c\) and \(2^c^{\infty}\) are not perfect.

**Theorem 3.** \((2^b)^{\eta} = (2^b)^{\infty} = 2^r\). The spaces \(2^b\) and \(2^b^{\infty}\) are not perfect, where \(0 < r \leq 1\)

**Proof:** Since, \(2^b^{\infty} \subset 2^c \subset 2^{\infty}\)

By Lemma 1 (ii) and theorem 1,

\[ 2^r = (2^{\infty})^\eta \subset (2^b)^{\infty} \subset (2^b)^\eta \]

Then, in order to prove the theorem, it is sufficient to show that \((2^b)^\eta \subset 2^r\).

Let \((a_{mn}) \notin 2^r\).

Then we can find sequences \(< m^i >\) of natural numbers with \(m^0 = 1, n^0 = 1\) such that

\[ \sum_{m=1}^{m_i} \sum_{n=1}^{n_i} a_{mn} l^r > \frac{1}{(i+1)^{r/2}}, \quad i = 0, 1, 2, 3, \ldots. \]

Define a sequence \((x_{mn})\) by

\[ x_{mn} = (i+1) \cdot \frac{1}{3} \]

for \(m^i-1 < m \geq m^i\) and \(n^i-1 < n \leq n^i\), for all \(i \in \mathbb{N}\).

Then \((x_{mn}) \in 2^c^{\infty}\).

Now,

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} x_{mn} l^r = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} x_{mn} l^r = \infty \text{ (because } 0 < \frac{5r}{6} < 1) \]

Hence \((x_{mn}) \in 2^b^{\infty}\).

When \(k \leq n \leq n^{k+1}\) for all \(k = 1, 2, 3, \ldots.\)

Then

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( a_{mn} l^r + x_{mn} l^r \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} l^r + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn} l^r = \infty \]

**Theorem 4.** \((2^b)^{\eta} = 2^r\). The space \(2^b\) is not perfect.

**Proof:** Since, \(2^b = 2^c \subset 2^{\infty}\)

By Lemma 1 (ii) and theorem 1,

\[ 2^r = (2^{\infty})^\eta \subset (2^b)^{\infty} \subset (2^b)^\eta \]

conversely let \((a_{mn}) \in (2^b)^\eta\)
Then \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn} x_{mn}| r < \infty \) for all \((a_{mn}) \in 2\sigma\).

Define a sequence \((x_{mn})\) as
\[
x_{mn} = 1 - x_{m+1,n} - x_{m,n+1} \quad \text{for all} \ m, n \in \mathbb{N}
\]
Then \((x_{mn}) \in 2\sigma\).

and hence
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn} x_{mn}| r = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}| r < \infty
\]

\(\Rightarrow (a_{mn}) \in 2I_r\)

Therefore, \((2\sigma) \cap \eta \subset 2I_r\).

Thus, we get \((2\sigma) \eta = 2I_r\)

Furthermore,

\(\because\) since, \((2\sigma) \eta = \{[2\sigma] \} = \{[2I_r] \} = 2I^\infty \neq 2\sigma\)

Then, the space \(2\sigma\) is not perfect.

**Theorem 5.** \((2\sigma \cap 2I^\infty) \cap \eta = 2I_r\). The space \(2\sigma \cap 2I^\infty\) is not perfect where \(0 < r \leq 1\).

**Proof.** Since \((2\sigma \cap 2I^\infty) \subset 2I^\infty\)

By Lemma 1 (ii) and theorem (1),

\[2I_r = (2I^\infty) \eta \subset (2\sigma \cap 2I^\infty) \eta\]

conversely, Let \((a_{mn}) \notin 2I_r\)

\(\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}| r = \infty\)

Define a sequence \((x_{mn})\) as
\[
x_{mn} = 1, \quad \text{for all} \ m, n \in \mathbb{N}
\]
Then, \((x_{mn}) \in 2\sigma \cap 2I^\infty\)

But
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn} x_{mn}| r = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}| r = \infty
\]

\(\Rightarrow (a_{mn}) \notin (2\sigma \cap 2I^\infty) \eta\)

Hence \((2\sigma \cap 2I^\infty) \eta \subset 2I_r\)

Thus, we get \((2\sigma \cap 2I^\infty) \eta = 2I_r\)

Furthermore,

\[2I^\infty \cap \{2\sigma \cap L_r \} \eta = \{2I_r \} \eta = 2I^\infty \eta \notin 2\sigma \cap 2I^\infty\]

\(\Rightarrow\) The space \(2\sigma \cap 2I^\infty\) is not perfect.

**Remark:** From theorem (1), (2), (3), (4), (6). It is obvious that

\[\{2I_r \} = \{2I^\infty \} = \{2I_r \} \eta = \{2I^\infty \} \eta = 2I^\infty \eta \neq 2\sigma\]

**REFERENCES**

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